PRISMS AND PYRAMIDS: Sequence Overview

Summary of learning goals

This sequence provides an accessible context for students to use simple algebra. Students build their skills in algebra by developing algebraic rules for the numbers of faces, edges and vertices in prisms and pyramids. They make deductions about unknown prisms and pyramids from these rules (links to equation solving) and then use their algebraic expressions to show that Euler's Formula works for all prisms and pyramids. Students also build their spatial skills through construction of pyramids and prisms.

Australian Curriculum: Mathematics (Year 7)

ACMNA175: Introduce the concept of variables as a way of representing numbers using letters.

- Understanding that arithmetic laws are powerful ways of describing and simplifying calculations and that using these laws leads to the generality of algebra.

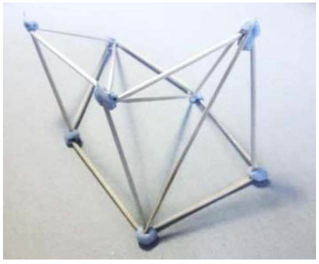
Summary of lessons

Who is this Sequence for?

This stand-alone lesson is designed for students who are beginning to learn algebra. Students commencing this lesson will need to understand that pronumerals stand for numbers and know the most basic conventions of algebra. For the Euler's formula task, students also need be able to collect like terms, e.g. they will need to be confident that $2b \neq b + 2$ and be able to simplify expressions such as $(b + 2) - 2b$. The lesson can involve solving very simple equations.

Lesson 1: Let's Face It

Students are presented with the challenge of finding the number of faces, edges or vertices of a pyramid or prism given the number of sides of the base shape. They explain the relationships between these numbers and describe them with algebraic expressions. Students then use the expressions to find prisms and pyramids with specified numbers of faces (etc.) and also use the expressions to verify Euler's Formula for these 3D shapes. This may be students' first encounter with proving using algebra. They test whether Euler's Formula holds for regular and irregular polyhedra of students' own construction.



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Reflection on this sequence

Rationale

A focus on properties of familiar 3D shapes has been selected as the vehicle for describing relationships with algebra. Students work with concrete objects, number patterns, verbal expressions and algebraic expressions of the relationships that they find. Together, all these representations provide meaning and purpose for the algebra. Application of their algebraic rules to Euler's Formula could be a first opportunity for students to prove a result using algebra.

reSolve Mathematics is Purposeful

This lesson emphasises the links between algebra and the physical world, in this case 3D shapes. Students develop skills in reasoning and communication as they make and justify generalisations, verbally and algebraically. The geometric situation illustrates the meaning of the algebra, and the algebra can describe the geometry and make predictions about it. Students develop a sense of why algebra is useful.

reSolve Tasks are Challenging Yet Accessible

Using constructed prisms and pyramids as a vehicle for developing algebraic understanding provides students with an accessible entry point to generalisation. There is an opportunity to link formal equation solving processes with intuitive methods.

Students are challenged to manipulate algebraic variables to find unknowns, and to produce two short algebraic proofs.

reSolve Classrooms Have a Knowledge Building Culture

The lesson is a guided investigation in which students can work together to discover patterns and make deductions. Each task builds on the previous task to help students move toward a greater understanding of what algebra can do.

We value your feedback after this lesson via http://tiny.cc/lesson-feedback

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PRISMS AND PYRAMIDS

Lesson 1: Let's Face It

Australian Curriculum: Mathematics (Year 7)

ACMNA175: Introduce the concept of variables as a way of representing numbers using letters.

- Understanding that arithmetic laws are powerful ways of describing and simplifying calculations and that using these laws leads to the generality of algebra.

Lesson abstract

Students are presented with the challenge of finding the number of faces, edges or vertices of a pyramid or prism given the number of sides of the base shape. They explain the relationships between these numbers and describe them with algebraic expressions. Students then use the expressions to find prisms and pyramids with specified numbers of faces (etc.) and also use the expressions to verify Euler's Formula for these 3D shapes. They test whether Euler's Formula holds for regular and irregular polyhedra of students' own construction.

Mathematical purpose (for students)

Algebra can be used to describe relationships, to make predictions and to show that relationships are always true.

Mathematical purpose (for teachers)

The main purpose is for students to use simple algebra in a meaningful context. Algebra is used in three modes. First, students find patterns in the numbers of faces, vertices and edges and express them algebraically. Then they can use these algebraic expressions in an equation solving mode to identify mystery shapes. In the additional task, students use algebra in a generalisation mode to prove that Euler's formula is true for all prisms and pyramids. Students will incidentally build their spatial skills through construction of pyramids and prisms.

Lesson Length **60 -90 minutes**

Vocabulary Encountered

- **pyramid**
- **prism**
- **faces**
- **edges**
- **vertices**
- **polyhedron**
- **Euler's Formula**

Lesson Materials

- **at least one prism and one pyramid for display**
- **1 x class set Geoshapes or other materials (e.g. Blu-Tak; toothpicks or straws; scissors)**
- **Student Sheet 1 – Prisms (1 per student)**
- **Student Sheet 2 – Pyramids (1 per student)**
- **Student Sheet 3 – Mystery 3D Shapes (1 per student)**
- **Student Sheet 4 – Euler's Formula (optional, 1 per student)**
- **soccer ball (optional)**

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What are prisms and pyramids?

Show some examples of prisms and pyramids and review the definitions and simple properties, for example:

> The Great Pyramid of Giza is a square pyramid. The base is a square, so the pyramid has four triangular sloping faces. The faces all meet at one point at the top. This pyramid has 5 vertices, 5 faces (including the base on the ground), and 8 edges.

Photographer: Nina Aldin Thune https://commons.wikimedia.org/wiki/File:Kheops-Pyramid.jpg

By Bogdan Costea (http://openclipart.org/media/files/bogdanco/3099) [CC0], via Wikimedia Commons

This cardboard box is a rectangular prism. The base is a rectangle (4 sides), so there are four vertical faces. This prism has 8 vertices, 6 faces, and 12 edges. The vertical faces of all prisms have four edges with both pairs of edges parallel. The top of a prism is the same shape as (congruent to) the base and parallel to it.

Investigating Pyramids and Prisms with Algebra

This section has two main tasks:

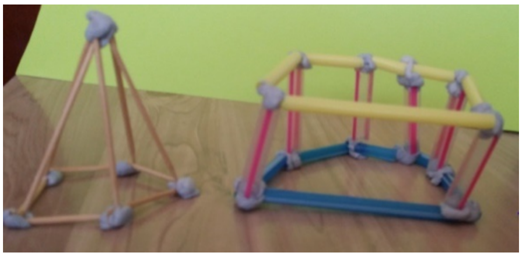
1. Finding the rules that give the number of vertices, edges and faces of a prism or pyramid if the shape of the base is known, and expressing these rules using algebra (see Student Sheet 1 – Prisms and Student Sheet 2 – Pyramids).
2. Using the rules to find unknown quantities and make judgements about the 3D shapes (see Student Sheet 3 – Mystery 3D Shapes).

Finding the rules

The best strategy for most students is to construct the 3D shapes, gather data, then look for patterns and reasons.

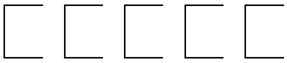
Students can construct models of prisms and pyramids using Geoshapes or similar, and count the faces, edges and vertices. Alternatively, they can use Blu-Tak for vertices and toothpicks or straws for edges, cutting to required lengths with scissors. The models do not have to be carefully constructed to see the patterns, and irregular prisms and pyramids will give the same results.

Students can record their results on Student Sheet 1 – Prisms and Student Sheet 2 – Pyramids.



Enabling Prompt

- Suggest that students think about how many edges a prism has for each side of the base. For example a pentagonal prism could be constructed from 5 sets of 3 edges as shown below:



Extending Prompt

- Amy made a pyramid and Ben made a prism. When they counted the number of edges they found that they had the same number. What shapes might they have made?
  (Ans: Any solids where the ratio of number of prism base sides to number of pyramid base is 2:3, e.g. a rectangular prism (4 sides) and a hexagonal pyramid (6 sides) both have 12 edges.)

Teacher Notes

- Students who progressively increase the number of sides on the base to see what happens to the number of faces, edges and vertices of the 3D shape may notice patterns that are less relevant to the investigation. For example, they may notice that e increases by 3 each row and write this addition in an incorrect rule like $e = b + 3$ rather than the multiplicative $e = 3b$. Encourage students to check that their rules fit their data and make physical sense, and review algebraic notation as required.

Number of sides of base (b)	*Number of Faces (f)*	*Number of Edges (e)*	*Number of Vertices (v)*
4	6	12	8
5	7	15	10
6	8	18	12
7	9	21	14

Expected Student Response

Property	***Rule in words***	***Why it works***	***Rule in algebra***
Number of Faces of Prism	Two more than the number of base sides	There is one face for each base side, then a top face and a bottom face.	$f = b + 2$
Number of Edges of Prism	Three times the number of base sides	The top and bottom faces each have b edges, and there is one vertical edge for each base side.	$e = 3b$ OR $e = b\times3$ OR $e = b + b + b$
Number of Vertices of Prism	Two times the number of base sides	There is one vertex for each side of the top face and one for each side of the bottom face.	$v = 2b$ OR $v = b\times2$ OR $v = b + b$
Number of Faces of Pyramid	One more than the number of base sides	There is one face for each base side, then a bottom face.	$f = b + 1$
Number of Edges of Pyramid	Two times the number of base sides	The bottom face has b edges, and there is one slanted edge for each base side.	$e = 2b$ OR $e = b\times2$ OR $e = b + b$
Number of Vertices of Pyramid	One more than the number of base sides	There is one vertex for each side of the bottom face plus one extra vertex at the top.	$v = b + 1$

Using the rules

Student Sheet 3 – Mystery 3D Shapes provides a chance for students to use the rules that they have just discovered. Some students may work with their rules in words rather than in algebra. Encourage linking algebra and the physical reality that the variables represent. This task can provide an opportunity for simple equation solving.

Warm Up

- How many vertices does a pyramid with a 12-sided base have? (Ans: 13 vertices)
- If a prism has 40 faces, how many sides on its base? (Ans: 38 sides)
- A pyramid has 14 edges. How many vertices does the pyramid have? (Ans: 7 sides, 8 vertices; this will probably be done in two steps, first finding how many sides are on the base.)

Mystery 3D Shapes

Ask students to write in their rules and then use them to fill in the blanks in the table. The first four rows are similar to the warm up questions. The last seven rows will require some extra reasoning or trial and error to work out whether the 3D shape is a pyramid or a prism (or impossible).

Expected Student Response

Prism or Pyramid?	***Number of sides of base (b)***	***Number of Faces (f)***	***Number of Edges (e)***	***Number of Vertices (v)***	***Teacher Notes***
Prism	20	22	60	40	
Prism	18	20	54	36	
Pyramid	16	17	32	17	
Pyramid	11	12	22	12	
Prism			22		Impossible: 22 is not a multiple of 3 and a prism's number of edges is of the form $3b$.
Prism	9	11	27	18	Since the number of edges is a multiple of 3 but not a multiple of 2, the shape is a prism.
Pyramid	10	11	20	11	The number of faces is the same as the number of vertices so this must be a pyramid.
Prism	50	52	150	100	The number of vertices is twice the base sides so this is a prism.
Pyramid	50	51	100	51	The number of faces is one more than the base sides so this is a pyramid.
Prism	8	10	24	16	The number of edges is a multiple of both 2 and 3, so students will need to look at the other columns. For a prism $3b=24$ giving $b=8$ and $f=10$. For a pyramid $2b=24$ giving $b=12$ and $v=13$.
Pyramid	12	13	24	13	

Additional Task: Euler's Formula

This task (Student Sheet 4 – Euler's Formula) could be a first opportunity for students to prove a result using algebra. Students will need to be familiar with collecting like terms.

Euler's Formula applies to all polyhedra that have no 'holes' going right through them. There is a proof, written for teachers or advanced students, at https://plus.maths.org/content/eulers-polyhedron-formula.

Students first calculate $f + v - e$ for some of the prisms and pyramids they have worked with, and then prove algebraically that $f + v - e = 2$ for all prisms and pyramids.

Enabling Prompt

- The expression $f + v - e$ cannot be simplified further. But if the formulas for f, v and e are used, then it is possible to collect like terms.

Expected Student Response

Proof for prisms:

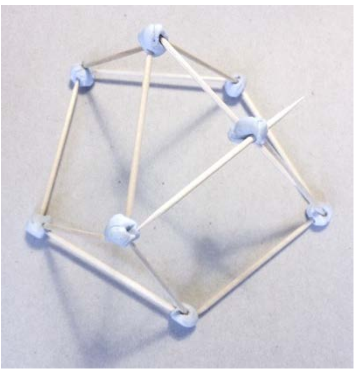
$$f + v - e = (b + 2) + (2b) - (3b)$$
$$= b + 2b - 3b + 2$$
$$= 2$$

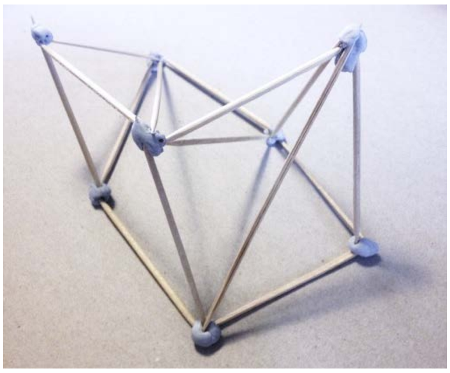
Proof for pyramids:

$$f + v - e = (b + 1) + (b + 1) - (2b)$$
$$= b + b - 2b + 1 + 1$$
$$= 2$$

Students will next use Euler's Formula to find unknowns for an icosahedron and a dodecahedron. It is worth noting that Euler's Formula works for both regular and irregular polyhedra (examples can be found at the links above).

Finally, students check that Euler's Formula works for a soccer ball (which is like a polyhedron but inflated) and for a polyhedron of their own construction. Encourage students to create irregular polyhedra like those pictured below to 'test the limits' of Euler's Formula. Students may find these polyhedra easier to construct.





re(Solve) MATHS BY INQUIRY

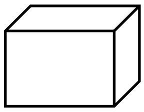
Investigating Prisms

Name: ______________

By Bogdan Costea
http://openclipart.org/media/files/bogdanco/3099) [CC0], via Wikimedia Commons

This cardboard box is a rectangular prism. The base is a rectangle. There are four vertical faces with parallel edges. The top of the box is the same shape as the base. There are 8 vertices, 6 faces, and 12 edges.

Your task is to find a way to predict the number of vertices, faces, and edges of a prism if you know the shape of its base. To help you find the rules construct some prisms and record their properties in the table below.

Sketch of prism	***Sketch of base***	***Number of sides of base (b)***	***Number of Faces (f)***	***Number of Edges (e)***	***Number of Vertices (v)***	***Prism name***
		4	6	12	8	Rectangular prism

Explain the rules and why they work, then write the rules using algebra.

Property	***Rule in words***	***Why it works***	***Rule in algebra***
Number of Faces of Prism			$f =$
Number of Edges of Prism			$e =$
Number of Vertices of Prism			$v =$

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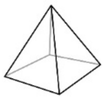
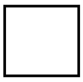
Investigating Pyramids

Name: ______________

Photographer: Nina Aldin Thune
https://commons.wikimedia.org/wiki/File:Kheops-Pyramid.jpg

The Great pyramid of Giza is a square pyramid. The base is a square. The pyramid has four triangular sloping faces. The faces all meet at one point at the top. There are 5 vertices, 5 faces (including the base on the ground), and 8 edges (4 sloping, 4 on the ground).

Your task is to find a way to predict the number of vertices, faces, and edges of a pyramid if you know the shape of its base. To help you find the rules construct some pyramids and record their properties in the table below.

Sketch of pyramid	***Sketch of base***	***Number of sides of base (b)***	***Number of Faces (f)***	***Number of Edges (e)***	***Number of Vertices (v)***	***Pyramid name***
		4	5	8	5	Square-based pyramid

Explain the rules and why they work, then write the rules using algebra.

Property	***Rule in words***	***Why it works***	***Rule in algebra***
Number of Faces of Pyramid			$f =$
Number of Edges of Pyramid			$e =$
Number of Vertices of Pyramid			$v =$

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Mystery 3D Shapes

Name: ______________

Write the rules you have already discovered here:

	Number of Faces	***Number of Edges***	***Number of Vertices***
Prism			
Pyramid			

Using these rules, fill in the blanks in the table below. You may need extra working out space. For the last six shapes you will need to work out whether they are prisms or pyramids.

Prism or Pyramid?	***Number of sides of base (b)***	***Number of Faces (f)***	***Number of Edges (e)***	***Number of Vertices (v)***
Prism	*20*			
Prism		*20*		
Pyramid				*17*
Pyramid			*22*	
Prism			*22*	
			27	
		11		*11*
	50			*100*
	50	*51*		
		10	*24*	
			24	*13*

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Euler's Formula

Name: ______________

Find the value of $f + v - e$ for at least ten of the solids (prisms and pyramids) you have worked with.

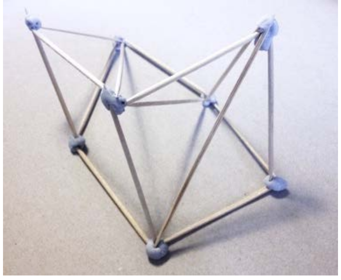
- What do you notice?
- Show that this relationship is true for all *prisms* using algebra.
- Show that this relationship is true for all *pyramids* using algebra.

A polyhedron is a 3D shape made by joining polygons. Euler's Formula states that for nearly any polyhedron $f + v - e = 2$. Polyhedra with 'holes' through them do not follow Euler's Formula.

- An icosahedron has 20 faces and 12 vertices. How many edges does it have?
- A dodecahedron has 12 faces and 30 edges. How many vertices does it have?

A soccer ball is made of pentagons and hexagons. It is not strictly a polyhedron (the pentagons and hexagons are not flat because the ball has been inflated) but we can imagine what it would look like with flat surfaces and apply Euler's formula.

- Count the number of faces, edges and vertices and check that Euler's formula works for a soccer ball.



Construct your own polyhedron and confirm that Euler's Formula works for it. It can be irregularly shaped like the example in the picture.

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