re(Solve) MATHS BY INQUIRY

What you need to know: VARIATION

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reSolve: Mathematics by Inquiry is funded by the Australian Government Department of Education and Training. Managed by the Australian Academy of Science in collaboration with the Australian Association of Mathematics Teachers.

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"If there was no variation, there would be no need for statistics" (Snee, 1999).

It is hard to imagine what statistics would be like without variation. When we think about what is involved in statistical thinking, variation is key every step of the way.

What is variation?

Statistical variation refers to the nature of data to not always be the same. It has been defined by different authors as:

- "the quality of an entity (a variable) to vary" (Makar & Confrey, 2005)
- "How 'spread out' the data are", which becomes apparent through graphical display and data summaries (Watson, Fitzallen & Carter, 2013)
- The presence of uncertainty in data, so that, for example, "repeated measurements of the same quantity yield varying results." (Moore, 1990)

Statistical variation is central to the field of statistics as we know it.

Why does variation matter?

Variation forms an implicit part of any statistics curriculum, though it is rarely labelled as such.

Variation and statistical thinking

Moore (1990) was one of the first statisticians to argue for the centrality of variation in statistics. He proposed five core elements of statistical thinking:

1. "Omnipresence of variation in processes": being aware that everything involves some degree of variation and uncertainty
2. "Need for data about processes": having some question that can be answered through the collection of data
3. "Design of data production with variation in mind": designing a method of data collection with an awareness of uncontrolled variation and the introduction of planned variation
4. "Quantification of variation": describing random variation mathematically
5. "Explanation of variation": finding the systematic effects underlying the observed variation.

Variation takes central stage in four of these five steps: variation establishes the *need* for statistics, the *pursuit* of statistics, the *production* of statistics and the *interpretation* of statistics.

The American Statistical Association's Guidelines for Assessment and Instruction in Statistics Education (GAISE) Report provides a similarly aligned set of components for statistical problem solving (Franklin et al., 2007):

1. Formulating questions: anticipating variability
2. Collecting data: acknowledging variability and designing for difference
3. Analysing data: accounting of variability in data
4. Interpreting results: allowing for variability

It is clear that an understanding of the role of variability in statistical contexts is a critical component of statistical reasoning (Shaughnessy et al., 2004).

Variation and statistical inference

In the Australian Curriculum: Mathematics, inference is often presented as the final goal of statistical investigation. In Foundation, the Statistics & Probability strand focuses on asking questions and making inferences; by Year 10, students are expected to "make comparisons and draw conclusions" with a range of different data and graphical representations.

Makar & Rubin (2009) identified three key principles of statistical inference:

1. Generalising beyond the data
2. Using the data as evidence for generalisations
3. Employing probabilistic language to describe the generalisations

The third of these is elaborated as "Articulating the *uncertainty* embedded in an inference". This refers explicitly to putting into words the *variation* present in the data. Making statistical inferences requires students to look at their data sample and estimate the amount of uncertainty (or **variation**) in their findings, in order to make statements and conjecture a possible "true value" for a population (Wills 2017).

Common misconceptions

Without a stable foundation in variation, students (and teachers) become more vulnerable to common statistical misconceptions and fallacies, among these the Gambler's Fallacy and the Law of Small Numbers. Any of these misconceptions can have a lasting impact on a student's understanding of statistical analysis.

The Law of Small Numbers

The Law of Large Numbers is a key theorem in probability and statistics which states that a sufficiently large sample is representative of its population. The Law of Small Numbers is its fallacious counterpart, which (falsely) states that even a small sample should be highly representative of a population (Tversky & Kahneman 1974). Consider the following question (Tversky & Kahneman 1974, p. 1125):

> A town has a large hospital and a small hospital.
>
> About 45 babies are born each day in the large hospital. About 15 babies are born each day in the small hospital.
>
> For a year, each hospital records every day that more than 60% of the babies born in that hospital are boys.
>
> Which hospital do you think recorded more such days?
>
> A. The larger hospital
> B. The smaller hospital
> C. Both hospitals recorded about the same number of days

Statistically B is the most likely answer, as smaller samples tend to vary more from the mean. Answering C suggests a belief in the Law of Small Numbers.

Belief in the Law of Small Numbers has been held responsible for, amongst other things, the hiring and firing of hedge fund managers (Baquero & Verbeek, 2006), the difficulty of replicating the findings of psychological research experiments (Tversky & Kahneman, 1974), the popularity of lotteries (Rabin, 2000), and the unreliability of election polls (Galen, 2016). It stems from a flawed understanding of statistical variation, and, in particular, "the relation between sample size and sampling variability" (Tversky & Kahneman 1974, p. 1130).

The Gambler's Fallacy

The Gambler's Fallacy is linked to the Law of Small Numbers. This is the reasoning that given equally likely, independent, random possibilities, if the same result has occurred more times than expected, then it will occur fewer than expected in the future. It is best illustrated with the following classic example:

> A coin is flipped three times with the result HHH.
> Do you:
> A. bet on the fourth flip being H
> B. bet on the fourth flip being T
> C. choose a bet at random, as they are equally likely

To someone operating under the Gambler's Fallacy, B is the correct choice as the occurrences need to "balance out". However, because each flip of the coin is an independent event, the odds of heads/tails do not change between flips.

What does highlighting variation look like?

Variation in the Australian Curriculum: Mathematics is most often explicitly referred to in the context of outcomes of random processes, as in ACMSP067 "Conduct chance experiments, identify and describe possible outcomes and recognise variation in results" and ACMSP145 "conducting repeated trials of chance experiments, identifying the variation between trials and realising that the results tend to the prediction with larger numbers of trials" (i.e. the law of large numbers).

Of course, variation is present throughout the rest of the curriculum as well: any mention of making inferences, evaluating displays, or interpreting data implicitly requires students to pay attention to the variation in their data. But how can this implicit requirement become an explicit acknowledgement of variation?

Measures of central tendency and of spread

Both the Australian Curriculum and maths teachers often have an explicit focus on measures of central tendency over measures of spread (Shaughnessy, 1997). The focus on measures of central tendency leads to a model of statistics focused on noticing similarity, rather than difference: students are asked to identify categories with the *most* or *fewest* objects, design surveys to find the *most popular* result, and, in Years 7-10, find the *mean*, *median*, and *mode* of datasets. This focus may be partially due to the fact that standard deviation is the most commonly recognised measure of spread and is not introduced into the curriculum until Year 10 (Shaughnessy 1997, p.11)—however, graphic representations of data make the spread of data visible, and well-designed questions can draw attention to spread, without relying on complex formulations.

Consider an 8-sided die.

Questions emphasising measures of central tendency:

- What is the probability of rolling an 4?
- What is the probability of rolling a number higher than 6?

Questions emphasising variation:

- If you rolled the dice 40 times, how many times might you get a 6?
- What number of sixes would surprise you?
- If you rolled the dice in 3 sets of 40 rolls, how many times might you get a 6 in each set?

A curriculum with variation at the core

Watson (2005) has proposed a statistics curriculum built around Expectation and Variation. Under this curriculum, instead of asking students only "*What do you* expect *to happen?*", which places focus on measures of central tendency, the question is followed with "*Why might it not happen? What might happen instead? Would we be surprised?*". Instead of focusing on what is most likely to happen, students are asked to consider all of the possible outcomes.

When should variation be introduced?

Watson & Kelly (2002) found that students as young as Year 3 were able to understand and work with statistical variation. Moore (1990, p.144) argues that students should be introduced to evaluations of variation as early as possible because it structures their relationship with probability and statistics :

> "students who begin their education with spelling and multiplication expect the world to be deterministic; they learn quickly to expect one answer to be right and others wrong, at least when the answers take numerical form. Variation is unexpected and uncomfortable…the ability to deal intelligently with variation and uncertainty is the goal of instruction about data and chance."

A framework highlighting variation

The GAISE report outlines an accessible framework for statistics education with a highlight on variation (Franklin et al. 2007, pp.14-15). Interesting features of this framework include:

- A focus on noticing difference and *designing for* difference
- Explicit discussion of data distribution throughout all levels
- Questions are formulated with an emphasis on *comparing values* as well as *identifying the values with the most objects*. For example, asking "Are the words in a chapter of a fifth-grade book longer than the words in a chapter of a third-grade book?" rather than "Which grade has books with the longest words?"
- Explicitly labelling different types of variation, including:
    - measurement variation, the variation in repeated measurements of the same object
    - sampling variation, variation between different samples of the same population
    - induced variation, variation that comes from introducing different factors into an experiment
    - chance variation, variation that emerges through random processes. This is the form of variation that most often appears explicitly in curriculums and classroom.

Useful resources

Guidelines for Assessment and Instruction in Statistics (GAISE) Education: A Pre-K-12 Curriculum Framework

By Christine Franklin, Gary Kader, Denise Mewborn, Jerry Moreno, Roxy Peck, Mike Perry & Richard Scheaffer

Published by the American Statistical Association.

Top Drawer Teachers: Statistics

By Jane Watson, Noleine Fitzallen & Pauline Carter

https://topdrawer.aamt.edu.au/Statistics

Learning progression

Foundation to Year 2	
What students will come to know and understand	Things can be classified according to their attributes. This allows us to compare how they are the same and different. Things can be classified in different ways depending on the attribute of interest. When items are classified into categories, these categories can be quantified, and comparisons can be made between categories. Categorical data can be represented in different ways. Data can be represented visually to communicate information. Data can be used to make predictions and inferences. We can explore our expectations by collecting data: context and experience help us learn what to expect. Different words can be used to describe the likelihood of events.
Indicators of understanding	Students can name and use attributes to sort a collection. They use attributes to make comparisons between objects. Students recognise that one object has multiple attributes. They use different attributes to categorise items in multiple ways. Students count to determine if categories have more, fewer or the same number of elements. They use the count to make simple inferences. Students create visual representations that effectively communicate data. Students use data and representations to make simple inferences. Students ask and answer questions about categorical data by drawing on visual representations. Students use appropriate language to describe the likelihood of events.

Year 3 to Year 6	
What students will come to know and understand	Expectations and predictions can be explored by collecting data. Using visual representations to show the distribution of data allows us to see the variation in the data and informs our expectations. Variation in data can be quantified in different ways. Variation can be summarised to form inform expectations. Because of variation, the outcomes observed may not match predictions and expectations. Uncertainty can be quantified to communicate expectations (probability). Multiple repetitions and larger samples allow for regularities and patterns to be observed. The middle of the ordered data expresses what is typical. The mean indicates the balance point of a numerical data set. A simple random process means all possibilities have an equal chance. Distributing data to create “fair shares” can help to find the mean and variability about it.
Indicators of understanding	Students quantify and categorise data in a way that allows for comparisons. Students draw on observed regularities and patterns in data to form and justify expectations. Students explain whether experimental outcomes align with predictions and expectations and why this might be. Students collect and use data to answer questions and to explore their expectations and predictions. Students interpret and refer to visual representations of data to form inferences. Students describe distributions referring to clumps, clusters, gaps and typical values in the data. Students move between and interpret different representations of the same data set.

Year 7 to Year 8	
What students will come to know and understand	Data are numbers with a context – they tell a story. Data can be represented in different ways for different purposes. This allows for different comparisons and different inferences to be made. Different sampling methods produce samples with different biases. The middle of the data expresses what is typical. The middle of a data set can be quantified in different ways (mean, median and mode) for different purposes. The mean is the balance-point of a numerical data set. The median is the midpoint of the numerical data set. The mode is the most common number in a data set. The range quantifies the variation in the data. Context is important when interpreting inferences made from data. Uncertainty can be quantified as relative frequency.
Indicators of understanding	Students choose between different data representations to communicate a particular story and justify why one representation might be "better" than another. Students consider how context affects the way data are organised and interpreted. Students choose from different sampling methods to produce representative and random samples of different populations. Students explain the differences among mean, median and mode as measures of central tendency. They can determine the mean, median and mode of a numerical data set. Students recognise the distinction among measures of central tendency and identify the circumstances when each is most useful. Students reason proportionally with data to make inferences. Students create graphs to tell the stories in given data sets. Students make inferences from samples acknowledging uncertainty.

Year 9 to Year 10	
What students will come to know and understand	Data can be represented in different ways for different purposes. This may allow for different comparisons to be made and different conclusions to be drawn. Different sampling methods from the same data set produce samples representative of different populations. As the size of a sample increases so does the confidence in the story it tells. Relationships within and between variables can be quantified. Reasoning proportionally with data (e.g. two-way tables, box and whisker plots) allows for inferences and decisions to be made. Scatterplots can represent the relationship of two numerical variables visually. There are mathematical ways to express confidence in the data.
Indicators of understanding	Students draw conclusions from data visualisations that require proportional reasoning (e.g. two-way tables, box and whisker plots). Students determine appropriate sample sizes that will produce information representative of a population and apply appropriate techniques to get this sample (stratified and random sampling). Students can determine the best representation to use with a given data set in order to draw useful conclusions. Students can determine which type of ‘typical’ measure is appropriate to make sense of a data set. Students can use scatterplots, time series plots, and stem-and-leaf plots to make decisions on relationships, acknowledging uncertainty. Students can describe the uncertainty of their conclusions with confidence.

References

Australian Association of Mathematics Teachers. (2013). Top Drawer Teachers: Statistics. Retrieved May 3, 2019, from https://topdrawer.aamt.edu.au/Statistics

Australian Curriculum, Assessment and Reporting Authority. (2019). F-10 Curriculum: Mathematics. Retrieved May 6, 2019, from https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics

Baquero, G. & Verbeek, M. (2008). Do sophisticated investors believe in the law of small numbers?. Working paper. European School of Management and Technology.

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). Guidelines for Assessment and Instruction in Statistics (GAISE) Education: A Pre-K-12 Curriculum Framework. Alexandria, VA: American Statistical Association.

Galen, R. (2016). The Trump Uncertainty Principle". Retrieved May 30, 2019, from https://medium.com/the-american-singularity/the-trump-uncertainty-principle-bea8bcfff80b

Makar, K., & Confrey, J. (2005). "Variation-Talk": Articulating Meaning in Statistics. Statistics Education Research Journal, 4(1), 27-54.

Makar, K., & Rubin, A. (2009). A framework for thinking about informal statistical inference. Statistics Education Research Journal, 8(1), 82-105.

Moore, D. S. (1990). Uncertainty. In L. A. Steen (ed.), On the shoulders of giants: New approaches to numeracy (pp. 95-137). Washington, DC: National Academy Press.

Rabin, M. (2000). Inference by believers in the law of small numbers. Working paper. University of California at Berkeley.

Shaughnessy, J. M. (1997). Missed opportunities in research on the teaching and learning of data and chance. In F. Biddulph & K. Carr (eds.), People in mathematics education (vol.1 pp. 6-22). Waikato, New Zealand: Mathematics Education Research Group of Australasia.

Shaughnessy, J. M., Ciancetta, M. & Best, K. (2004). Students' Attention to Variability when Comparing Distributions. Paper presented at the research presession of the 82nd annual meeting of the National Council of Teachers of Mathematics, Philadelphia, PA.

Snee, R. D. (1999). Discussion: Development and Use of Statistical Thinking: A New Era. International Statistical Review, 67(3), 255-258.

Tversky, A. & Kahneman, D. (1974). Judgement under Uncertainty: Heuristics and Biases. Science, 185(4157), 1124-1131.

Watson, J. (2005). Variation and expectation as foundations for the chance and data curriculum. In P. Clarkson, A. Downton, D. Gronn & M. Horne (eds.), Building Connections: Theory, Research and Practice: Proceedings of the twenty-eighth conference of the Mathematics Education Research Group of Australasia. Melbourne: RMIT.

Watson, J. & Kelly, B. (2002). Can grade 3 students learn about variation?. In B. Phillips (ed.), Developing a statistically literate society? Proceedings of the 6th International Conference on the teaching of statistics, Cape Town. Voorburg: International Statistical Institute.

Wills, A. (2017). Ideas of statistical inference: Concepts. Retrieved May 22, 2019, from http://www.bristol.ac.uk/medical-school/media/rms/red/4_ideas_of_statistical_inference.html