re(Solve) MATHS BY INQUIRY

What you need to know: FRACTIONS

Contents

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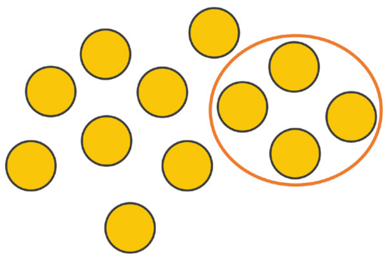
Students come to school with an intuitive fractional sense. Fractional concepts are linked to many everyday experiences, such as sharing between friends, measurement in cooking and time. Despite this, fractions are acknowledged as one of the most difficult and complex topics in mathematics to learn (Lamon, 2007), and to teach (Bobis, 2011). Smith (2002, p. 3) states that "no area of ... school mathematics is as mathematically rich, cognitively complicated, and difficult to teach".

What is a fraction?

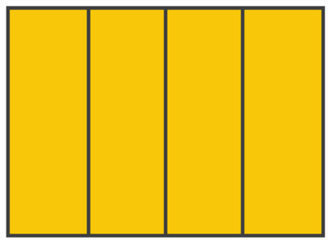
Fractions are rational numbers that express a multiplicative relationship between two quantities. These quantities may be discrete (countable items) or continuous (like area and length) (Australian Association of Mathematics Teachers, 2013). Because they express a relationship between two numbers, fractions are innately different to whole numbers and do not behave like counting numbers (Kieren, 1980; Siemon et al., 2011).

Some important definitions

In any fraction of the form $\frac{a}{b}$, *b* is the **denominator** and is the name or size of the parts. The **numerator**, *a*, represents the number of parts of that name or size.



A **discrete** model for fractions treats the parts as separate items. These separate items are countable units that do not get sub-divided.

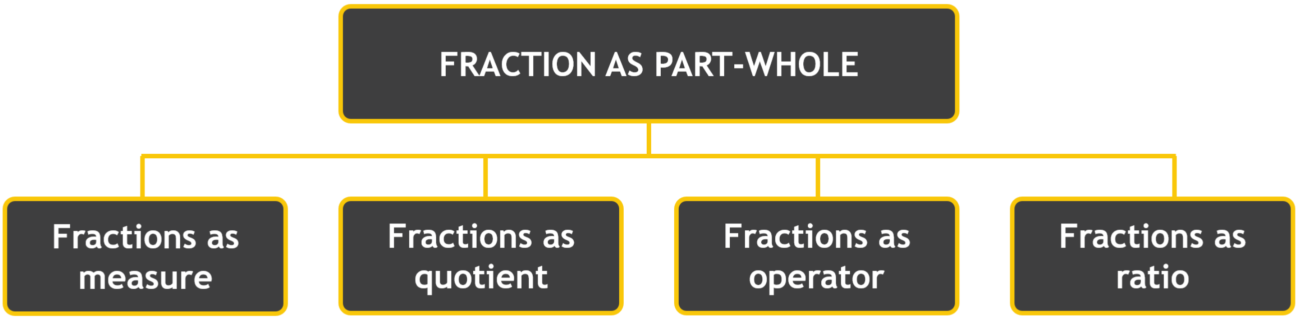


A **continuous** fraction model is also called an **area** model. This model can be continuously sub-divided into smaller parts.

Fractions represent multiple ideas and can be represented in different ways (Bobis, 2011). Adding It Up (2001), a study report on mathematics teaching by the National Research Council in the United States, says that a fraction is "not a single entity but has multiple personalities" (p. 233).

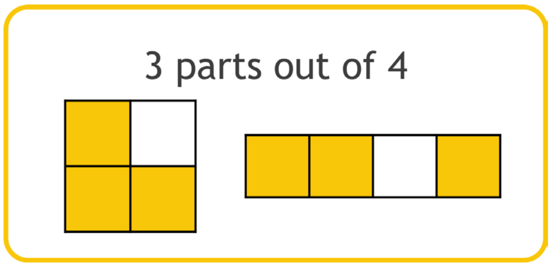
Kieren (1993) identified one overarching fraction construct, part-whole, with four sub-constructs: measure, quotient, operator, and ratio. To gain a strong understanding of fractions, teaching needs to emphasise the multiple meanings of fractions (Clarke, Roche, & Mitchell, 2008; Lamon, 2012; Van de Walle, Karp & Bay-Williams, 2015), and students need support to make connections between the different fractional constructs.

Sub-Constructs for Fractions



Fractions as part-whole

The fractions as part-whole construct expresses the relationship between the whole and the designated number of equal parts (Kieren, 1980). In this context, the denominator is the number of parts which make up the **whole**. The **part** is the number of equal parts that are taken together—the numerator. For example, $\frac{3}{4}$ is 3 parts out of 4 equal parts.

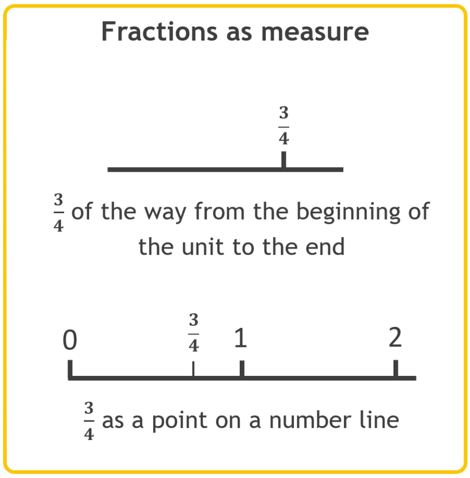


The part-whole construct for fractions permeates the other four fractional sub-constructs and so can be represented multiple ways.

Fractions as measure

Fractions can represent the measurement of a quantity, where the measure is relative to one unit of that quantity (Clarke, Roche & Mitchell, 2011). The number of equal parts in one unit can vary depending on how many times you partition the unit. Successive partitioning allows for greater precision when measuring.

Fractions as measure can be represented as a part-whole segment of a line—a point on a line where the line represents the whole or as a number line (Mitchell & Horne, 2011).



Under the fractions as measure construct, the focus is on "how much" rather than "how many parts" (van de Walle, 2015).

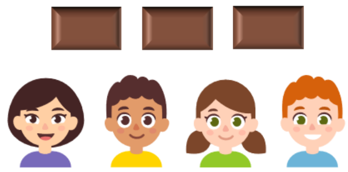
Fractions as quotient

The representation $\frac{a}{b}$ is equivalent to $a \div b$. This notion of fractions relates to the idea of "sharing equally". For example, if 3 chocolate bars are shared equally between 4 people, each person will receive $\frac{3}{4}$ of a bar. This can also be represented as $3 \div 4 = \frac{3}{4}$.

The fractions as quotient interpretation affords the opportunity to engage in renaming a fraction as a decimal or a percentage (Siemon et al., 2011). For example, $3 \div 4$ can be expressed as $\frac{3}{4}$ and 0.75.

$3 \div 4 = \frac{3}{4}$

3 shared between 4 is $\frac{3}{4}$ each



Fractions as operator

A fraction can be used as an operator. In this context fractions enlarge or reduce the size of a number. Operating with fractions challenges the misconception that multiplication always "makes bigger" and division always "makes smaller" (Clarke, Roche & Mitchell, 2008). For example, $12 \times \frac{3}{4} = 9$, whereas $12 \div \frac{3}{4} = 16$.

Fractions as operators

$12 \times \frac{3}{4} = 9$

$12 \div \frac{3}{4} = 16$

Fractions as ratio

Fractions can be used to express a ratio. A ratio compares the size of two sets of measures (Clarke, Roche & Mitchell, 2011). For example, in the image on the right, the number of green marbles is $\frac{3}{4}$ the number of blue marbles. The ratio of blue marbles to green marbles can be expressed as 6:8 and simplified to 3:4.

Fractions as ratios

3 : 4

The number of blue marbles is $\frac{3}{4}$ the number of green marbles

Another way to think about fractional sub-constructs is to consider *partition fractions* and *quantity fractions*, commonly used in Japanese classrooms (Yoshida, 2004). A *partition fraction* is an amount $\frac{a}{b}$ of an object partitioned into b equal parts and with a parts selected. For example, $\frac{1}{8}$ of a cake is a partition fraction. Partition fractions do not have universal units.

Quantity fractions are those with universal units. Fractions as measure are often quantity fractions. Examples include $\frac{1}{2}$ of a metre or $\frac{1}{2}$ as a number on a number line. Quantity fractions emphasise the importance of understanding *the whole*.

Substantial Mathematical Ideas

reSolve uses the term *Substantial Mathematical Ideas* to identify key understandings that are critical to the development of a mathematical concept. We have identified five features that make a mathematical idea substantial:

- CONCEPTUAL: Focuses on understandings over procedures and skills
- STRUCTURAL: Focuses on the structure of mathematics
- CONNECTED: Connects mathematical concepts across domains
- TRANSFORMATIVE: Challenges and repositions existing conceptions
- GENERATIVE: One idea generates another

Substantial Mathematical Ideas for fractions

Fractions express multiplicative relationships

Fractions express a ratio of part to whole, which is a multiplicative relationship.

The whole matters

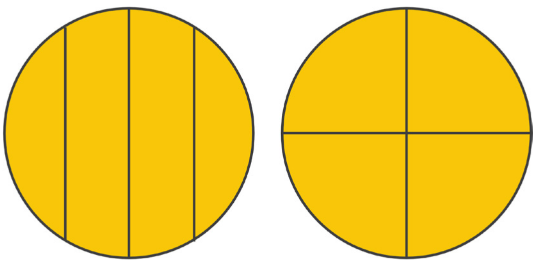
Fractions represent a ratio of the part to the whole (Fosnot & Dolk, 2002). It is critical that students understand that the size or amount of the whole determines the size of the parts. This means that $\frac{1}{2}$ o f one whole will not necessarily be the same as $\frac{1}{2}$ of another whole.

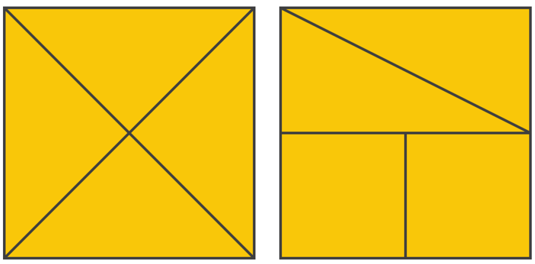
To compare, add or subtract fractions, a common whole is required (Jacob & Fosnot, 2007).

Fractional pieces need to be equal in size or amount

Fractions are not simply about the number or parts; fractions are concerned with *equal-sized parts*. Both of the circles below are divided into four parts, but only one circle is divided into quarters.

Fractional pieces do not need to be congruent in order to be equal in size when working with area models (Fosnot & Dolk, 2002). They can look different, yet still be equivalent. Both of the squares below have been divided into quarters—each fractional piece is equivalent in area.





For equivalence, the ratio must be kept constant

Equivalent fractions represent the same value or proportion of the whole. An understanding of equivalence will not come through rules such as 'what you do to the top, you must also do to the bottom'. Students need to appreciate that when fractions express the same ratio between numerator and denominator, they represent equivalent values (Fosnot & Dolk, 2002).

The properties of commutativity, associativity and distributivity hold true for fractions

The arithmetic properties of commutativity, associativity and distributivity apply to fractions (Jacob & Fosnot, 2007). As with integers, this forms the foundation of learning to operate with rational numbers.

Common misconceptions

Many students experience difficulties in working with fractions. Indeed, fractions pose one of the most challenging and complex mathematics topics to learn for students (Lamon, 2007). Fractions are written in a unique way and, as discussed, there are multiple fractional constructs that students need to understand. Many student misconceptions with fractions often arise from an over-generalisation of whole number knowledge (van de Walle et al., 2015). Students incorrectly think about and operate with fractions as whole numbers rather than relational numbers. Such misconceptions include:

Incorrect understanding of the denominator and numerator

Students fail to recognise a fraction as a number that expresses a multiplicative relationship between two numbers. Instead, they see the denominator and the numerator as separate values (Cramer & Whitney, 2010).

Not recognising the need for equipartitioning to form fractional parts

Students will focus on the 'number of parts' without appreciating that these parts need to be equal in size or amount. Students may also believe that fractional parts in area models need to be congruent and fail to recognise that parts can look different but still be equivalent in area.

Do not see the inverse relationship between the denominator and the size of the fractional parts

When working with fractions, the larger the denominator, the smaller the size of the pieces. This inverse relationship is not obvious to students if they see fractions as two unrelated whole numbers.

Misapplication of whole number operational procedures for whole numbers to fractions

To add and subtract fractions, the whole must be the same. This means that the whole will not change—the number of parts will.

Not recognising that the whole matters

Fractions express a relationship between the whole and the parts. One-half of one object can be very different to one-half of another object. To compare fractions, the whole must be the same. The concept of a common referent unit is often not appreciated by students.

The use of additive rather than multiplicative thinking

Fractions represent a multiplicative relationship. Students often apply additive strategies when working with fractions, e.g. finding equivalent fractions, operating with fractions.

Models for fractions

Typically students are provided with models and are expected to work with the pre-constructed representation. It is important for students to use a variety of representations to model fractions (Clarke, Roche & Mitchell, 2008). They also need experience in generating fractional models for themselves. When students generate their own models, they are provided with the opportunity to construct important concepts about fractions such as equipartitioning.

Fractional representations are classified into three groups.

Area

An area representation of fractions is a commonly used model, as it lends itself to equal shares and partitioning contexts. A whole can be partitioned into smaller parts, meaning there is an inverse relationship between the number of parts and the size of each part.

Students should have experiences working from the whole to the part, from the part to the whole, and from a part to a part.

Length

Length models are associated with fractions as measure. In real-world contexts this use of fractions is common, yet it is less common in classrooms. The number line is a powerful model that builds the understanding that fractions are numbers. It creates connections between whole numbers and fractions and is a helpful context to explore improper fractions (Mitchell & Horne, 2011).

Set

In this form of representation, the whole is recognised as a set of objects and equal sub-sets for the fractional parts (van de Walle et al., 2015). The set model connects to important real-world contexts relating to ratios.

Implications for practice

Students come to school with intuitive notions of sharing and partitioning which provides a useful starting point for the development of fractional concepts. Their experience of fractions is often related to sharing and measure. It is important that students are provided with multiple experiences and time to develop meaning of the different forms of fractions.

Clarke and colleagues (2008; 2011) offer some practical implications for teaching fractions:

1. Give greater emphasis to the meaning of fractions than on procedures for manipulating them
2. Emphasise that fractions are numbers and make extensive use of number lines
3. Focus on improper fractions and equivalences, including the flexible naming of fractions
4. Link fractions to key benchmarks and encourage estimation
5. Give emphasis to fractions as division
6. Provide a variety of models to represent fractions

Useful resources

Fractions: Teaching for Understanding

Edited by Jenni WCay & Janette Bobis

Published by Australian Association of Mathematics Teachers (AAMT)

Top Drawer Teachers: Fractions

By Jenni Way

https://topdrawer.aamt.edu.au/Fractions

Young Mathematicians at Work: Constructing Fractions, Decimals, and Percents

By Catherine Fosnot & Maarten Dolk

Published by Heinemann

Contexts for Learning Mathematics: Investigating Fractions, Decimals, and Percents

A series of units published by Heinemann:

- *Field Trips and Fund-Raisers: Introducing Fractions* by Catherine Fosnot
- *The Californian Frog-Jumping Contest: Algebra* by Bill Jacobs & Catherine Fosnot
- *The Mystery Meter: Decimals* by Bill Jacobs, John Michael Siegfried & Catherine Fosnot
- *Best Buys, Ratios, and Rates: Addition and Subtraction of Fractions* by Bill Jacob & Catherine Fosnot
- *Exploring Parks and Playgrounds: Multiplication and Division of Fractions* by Lynn Tarlow-Hellman & Catherine Fosnot
- *Minilessons for Operations with Fractions, Decimals, and Percents* by Kara Imm, Catherine Fosnot & Willem Uittenbogaard

Learning progression

Foundation to Year 1	
What students will come to know and understand	Fractions are used in everyday life. Halves are created when the whole of an object is divided into two equal-sized parts. An object can be divided into halves in multiple ways. The two halves of an object will always be the same size even though they may be different shapes. One half of one object may be different in size to one half of another object. The size of the whole is important.
Indicators of understanding	Students recognise and use ‘half’ in varied everyday contexts. Students divide an object into two equal-sized pieces and name each piece ‘half’. Students use direct and/or indirect comparison to determine whether an object has been divided in half or into two non-equal sized parts. Students refer to the size of the whole to explain why one half of one object might be different in size to one half of another object. Students rebuild a whole using two halves. Students visualise and/or estimate half of an object. Students use the language of ‘half’ in appropriate contexts.

Year 2	
What students will come to know and understand	Fractional parts need to be equal in size or amount. The number of equal parts names the fractional unit. There is an inverse relationship between the number of equal parts and the size of the parts. Fractions can be named and renamed in different ways. The size of the whole matters. The size of the unit matters. Fractions can be represented symbolically.
Indicators of understanding	Students divide a whole into equal-sized pieces to represent fractions. Students use direct and/or indirect comparison to determine whether an object has been divided in halves, quarters or eighths. Students recognise that parts do not need to be congruent to be equal in size. They create non-congruent fractional parts. Students apply different strategies for equipartitioning (e.g. repeated halving to make quarters and eighths). Students record fractions using words and/or symbolic notation. Students recognise that the more equal parts the whole is divided into, the smaller each part becomes. They use this to explain why larger denominators result in smaller parts. Students determine how many units of a fractional part are required to rebuild a whole. Students model and name the result of combining fractional parts (e.g. two quarters make one half, eight quarters makes two wholes).

Year 3	
What students will come to know and understand	Fractions express a multiplicative relationship between the part and whole. Equivalent fractions express the same amount by using different-sized fractional parts. For equivalence, the ratio must be kept constant. Fractions can be named and renamed in different ways. The size of the whole matters. The size of the unit matters.
Indicators of understanding	Students recognise that fractions represent a multiplicative relationship and explain the relationship between the part and whole. They realise that a fraction represents one number, not two separate numbers. Students connect factors and multiples to fractions and their representations (e.g. folding halves into thirds creates sixths). Students use visual representations to explore equivalent fractions. They recognise some familiar fractions as equivalent (e.g. $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{3}{6}$). Students use equivalence to rename familiar fractions. Students subdivide fractions to create new fractions (e.g. halving thirds to make sixths). They make connections between these subdivisions to equivalent fractions. Students recognise that parts do not need to be congruent to be equal in size. They create non-congruent fractional parts. Students apply different strategies for equipartitioning (e.g. halving thirds to make sixths).

Year 4	
What students will come to know and understand	Fractions express a multiplicative relationship between the part and whole. Equivalent fractions express the same amount by using different-sized fractional parts. For equivalence, the ratio must be kept constant. Fractions can be named and renamed in different ways. Fractions are numbers and can be represented on a number line. Part of a set of items can be represented as a fraction. Fractions represent division.
Indicators of understanding	Students connect factors and multiples to fractions and their representations (e.g. folding fifths into thirds creates fifteenths). Students recognise a multiplicative relationship between equivalent fractions. They connect equivalent fractions with factors and multiples. Students use multiplication and division to create equivalent fractions. Students use equivalence to rename fractions. Students connect fractions to division. They use visual representations to solve problems involving division and represent the result as a fraction. Students count forwards and backwards in fractions using a number line. They connect counting to the addition and subtraction of fractions with like denominators. Students use visual representations and informal mental strategies to find fractions of a set.

Year 5	
What students will come to know and understand	Fractions express a multiplicative relationship between the part and whole. Equivalent fractions express the same amount by using different-sized fractional parts. For equivalence, the ratio must be kept constant. Fractions can be named and renamed in different ways. Fractions are numbers and can be represented on a number line. Part of a set of items can be represented as a fraction. Fractions represent division. Unit fractions are fractions with a numerator of one. Unit fractions make it easier to compare fractions.
Indicators of understanding	Students represent division as a fraction. They use visual representations and mental strategies to solve problems involving division and represent the result as a fraction. Students develop informal strategies for the addition and subtraction of fractions with related units (denominators). Students estimate answers to questions involving the addition and subtraction of fractions with related units (denominators). Students estimate and justify the location of fractions and mixed numbers on the number line in relation to whole numbers and other fractions. Students use benchmarks (e.g. 0, 1/2, 1) to estimate the magnitude of fractions. Students compare and order fractions. Students develop and apply mental strategies for finding fractions of a set. Students use unit fractions to compare fractions.

Year 6	
What students will come to know and understand	Fractions express a multiplicative relationship between the part and whole. For equivalence, the ratio must be kept constant. Fractions can be named and renamed in different ways. Fractions are numbers. They can be represented on a number line. Part of a set of items can be represented as a fraction. Fractions represent division.
Indicators of understanding	Students represent division as a fraction. They use visual representations and mental strategies to solve problems involving division and represent the result as a fraction. Students apply strategies for the efficient addition and subtraction of fractions with related units (denominators). Students estimate answers to questions involving the addition and subtraction of fractions with related units (denominators). Students estimate and justify the location of fractions and mixed numbers on the number line in relation to whole numbers and other fractions. Students use benchmarks (e.g. 0, 1/2, 1) to estimate the magnitude of fractions. Students compare and order fractions. Students apply mental strategies for finding fractions of a set. Students apply knowledge of fractions to solve a variety of problems.

Year 7 and Year 8	
What students will come to know and understand	Fractions express a ratio between the part and whole. For equivalence, the ratio must be kept constant. Fractions can be named and renamed in different ways. Fractions are numbers and can be represented on a number line. Fractions can be used as operators. Fractions represent division.
Indicators of understanding	Students place positive and negative fractions onto a number line. Students compare and order positive and negative fractions. Students apply mental strategies for finding fractions of a set. Students develop and use informal strategies for the addition and subtraction of fractions. Students apply strategies for the efficient addition and subtraction of fractions. Students use visual representations to model and solve problems involving the multiplication and division of fractions. Students develop and use informal mental strategies to solve problems involving the multiplication and division of fractions. Students apply mental strategies to solve problems involving the multiplication and division of fractions. Students represent ratios as a fraction. Students apply knowledge of fractions to solve a variety of problems.

References

Australian Association of Mathematics Teachers. (2013). Top Drawer Teachers: Fractions. Retrieved May 30, 2019, from https://topdrawer.aamt.edu.au/Fractions

Bobis, J. (2011). Fractions: Best evidence and its implications for practice. In J. Way & J. Bobis (eds.), Fractions: Teaching for Understanding (pp. 11-20). Adelaide: AAMT.

Clarke, D., Roche, A. & Mitchell, A. (2008). Ten practical tips for making fractions come alive and make sense. Mathematics Teaching in the Middle Years, 13(7), 373-380.

Clarke, D., Roche, A. & Mitchell, A. (2011). One-to-one student interviews provide powerful insights and clear focus for the teaching of fractions in the middle years. In J. Way & J. Bobis (eds.), Fractions: Teaching for Understanding (pp. 23-32). Adelaide: AAMT.

Cramer, K. & Whitney, S. (2010). Learning rational number concepts and skills in elementary school classrooms. In D. V. Lambdin & F. K. Lester, Jr (Eds.), Teaching and Learning Mathematics: Translating Research for Elementary Schjool Teachers (pp. 15-22). Reston, VA: NCTM.

Fosnot, C. & Dolk, M. (2002). Young mathematicians at work: Constructing fractions, Decimals, and Percents. Portsmouth, NH: Heinemann.

Jacob, B & Fosnot, C. (2007). Best buys, Ratios, and Rates: Addition and subtraction of fractions. Portsmouth, NH: Heinemann.

Kieren, T. (1980). The rational number construct: Its elements and mechanisms. In T. E. Lieren (ed.), Recent Research on Number Learning (pp.125-149). Columbus, OH: Eric Clearinghouse for Science, Mathematics and Environmental Education.

Kieren, T. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), Rational Numbers: An Integration of Research (pp. 49-84). Hillsdale, NJ: Lawrence Erlbaum.

Lamon, S. (2007). Rational numbers and proportional reasoning: Towards a theoretical framework for research. In F. Lester (Ed.), Second Handbook of Research in Mathematics Teaching and Learning (pp. 692-666). Reston, VA: NCTM.

Lamon, S. (2012). Teaching Fractions and Ratios for Understanding: Essential Content Knowledge and Instructional Strategies. New York, NY: Taylor & Francis Group.

Mitchell, A. & Horne, M. (2011). Measurement matters: Fraction number lines abd length concepts are related. In J. Way & J. Bobis (eds.), Fractions: Teaching for Understanding (pp. 52-62). Adelaide: AAMT.

National Research Council. (2001). (J. Kilparick, J. Swafford, & B. Findell, Eds.). Adding it up: Helping Children Learn Mathematics. Washington, D.C.: National Academy Press.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R., & Warren, E. (2011). Teaching Mathematics: Foundations to Middle Years. South Melbourne: Oxford University Press.

Tarlow-Hellman, L. & Fosnot, C. (2007). Exploring Parks and Playgrounds: Multiplication and division of fractions. Portsmouth, NH: Heinemann.

Van de Walle, J., Karp, K. & Bay-Williams, J. (2015). Elementary and Middle School Mathematics: Teaching Developmentally, 9th Edition. Essex: Pearson Education.

Yoshida, K. (2004). Understanding how the concept of fractions develops: A Vygotskian perspective. In M. J. Hoines & A. B. Fugelstad (Eds.), Proceedings of the 28th conference of the International Group for the Psychology of Mathematics Education (pp. 473-480). Bergen, Norway.