PRIME FACTORISATION
Lesson 4: Stars

Australian Curriculum: Mathematics – Year 7

ACMNA149: Investigate index notation and represent whole numbers as products of powers of prime numbers.

- Solving problems involving lowest common multiples and greatest common divisors (highest common factors) for pairs of whole numbers by comparing their prime factorisation.

Lesson abstract

This lesson is a substantial investigation of the shapes formed by joining points evenly distributed around a circle. Students generate and test hypotheses and explore the connection between the HCF and LCM of the number of points on the circle and the jump size, and the shape produced. The lesson moves to then look at how prime factorisation can efficiently determine the HCF and LCM of two numbers. Students explore why the HCF (a,b) x LCM (a,b) = ab.

Mathematical purpose (for students)

To carry out a substantial mathematical investigation, by looking for patterns, making and testing hypotheses and writing generalisations.

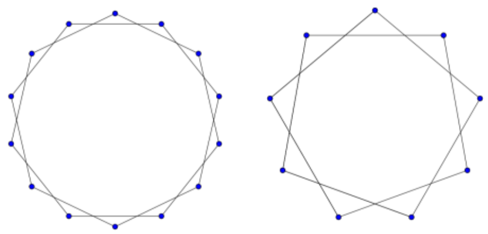
Mathematical purpose (for teachers)

The lesson provides a purposeful, rich and aesthetically pleasing activity that builds on the knowledge developed during the first two lessons related to prime factorisation. There are also smaller tasks to investigate the many connections between HCF and LCM and prime numbers.

Lesson Length — Two parts, each 60 minutes approximately.

Vocabulary Encountered

- Highest Common Factor (HCF)
- Lowest Common Multiple (LCM)



Lesson Materials

- Student Sheet 1 – Exploring Stars (1 per student)
- Student Sheet 2 – Circle Templates for Drawing Stars (1 per student)
- rope, string or yarn (approximately 15m)
- Student Sheet 3 – Exploring Stars Recording Sheet (1 per student)
- Student Sheet 4 – Finding Numbers Using HCF & LCM (1 per student)
- Student Sheet 5 – HCF & LCM Reflection (1 per student)

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science

Inquiry 1: Exploring Stars

It may be helpful to do this initially as a class physical activity before handing out Student Sheet 1 – Exploring Stars.

Seat 10 students in a circle. One student walks around the outside of the circle handing a rope (or string or yarn) to every second person to hold. The rope is continually passed around until it makes it back to the first person to hold it. A shape – sometimes a simple polygon but usually a star – is formed in the middle.

This makes it easy to see the shape formed and to count the number of laps around the circle.

This can be repeated passing the rope or ribbon to every third, fourth etc. person, observing the star shape made.

Students can then use computers or Student Sheet 2 – Circle Templates for Drawing Stars to explore the 'stars' made using different jump sizes around circles with different numbers of points.

After they have explored the stars for a while, discuss the shapes that are made. Jump size 1 always makes a simple polygon. Other jump sizes give simple polygons (e.g. a pentagon, heptagon) but most of the shapes look like stars, with varying numbers of points. Ask students to record information using Student Sheet 3 – Exploring Stars Recording Sheet. Being systematic with numbers will help highlight patterns.

Number of points around the circle	Jump size	Number of points on 'star' shape	Gap between points on 'star' shape	Number of laps around the circle	Total number of points passed (laps x points around circle)
10	2	5 (pentagon)	2	1	10
10	3	10	1	3	30
10	4	5	2	2	20
↓		↓			↓
14	2	7 (heptagon)	2	1	14
14	3	14	1	3	42
↓			↓		↓

Students will note that when the jump size is a factor of the number of points around the circle, the lines will join on the first lap of the circle.

Encourage the students to explore jump sizes that are not factors of the points around the circle. Some examples might be:

- 40 point circle with a jump size of 6
- 35 point circle with a jump size of 14
- 13 point circle with a jump size of 5
- 20 point circle with a jump size of 8 and also 15 point circle with a jump size of 6 (both pentagons)
- 84 point circle with a jump size of 22 (visualisation will need to replace drawing here)

The highest common factor (HCF) of the two numbers is equal to the gap between the points touched around the circle. For example:

- 35 point circle with a jump size of 14 will end up touching every 7^{th} point, making a 5-pointed star. 7 is the HCF of 35 and 14.
- 13 point circle with a jump size of 5 will end up touching every point, making a 13-pointed star. 1 is the HCF of 13 and 5.

The lowest common multiple (LCM) of the two numbers is the total number of points passed as you move around the circle. For example:

- In a 35 point circle with a jump size of 14, you need two complete laps of the circle. This is 70 points passed. 70 is the LCM of 35 and 14.
- In a 13 point circle with a jump size of 5, you need five complete laps of the circle. This is 65 points passed. 65 is the LCM of 13 and 5.

Teacher Notes - Extension Information

The star made using a 13 point circle with a jump size of 5 is called a $\frac{13}{5}$ star polygon. It has 13 points and there are 5 laps of the circle needed to make the star.

Note that since $\frac{35}{14} = \frac{5}{2}$, the star made using a 35 point circle and jump size of 14 is a $\frac{35}{14} = \frac{5}{2}$ star polygon.

Also, a $\frac{5}{3}$ star polygon is a $\frac{5}{2}$ star polygon done backwards.

(See http://mathworld.wolfram.com/StarPolygon.html)

The angle at each vertex of a regular star polygon can be found using the formula for the angle in a regular polygon $\theta = \frac{180(n-2)}{n}$ by substituting the fraction above for *n*.

For example, the angle at each vertex of a $\frac{5}{2}$ star polygon is $\frac{180(\frac{5}{2}-2)}{\frac{5}{2}} = 36°$

For the $\frac{5}{3}$ star polygon it is $\frac{180(\frac{5}{3}-2)}{\frac{5}{3}} = -36°$ indicating the star is produced in the opposite direction.

Computer resources

- The activity can be done interactively at https://nrich.maths.org/2669 or from Maths 300 – 'Hunting for Stars' (subscription). Nrich also provides printable circles with differing numbers of points marked on the circumference - http://nrich.maths.org/8506
- Circles with various numbers of points can be generated with Geogebra at https://ggbm.at/WMHwH5Te. Simply change the number of points using the slider, and copy the resulting arrangement into a printable document. The size of the circle can be changed by dragging the point at the top up or down.
- Alternatively, use Geogebra to draw the lines between the dots too. Go to https://ggbm.at/WMHwH5Te. Choose "open with Geogebra app" from the menu at the top right hand corner. Dial up the desired number of points, select the "segment tool" from the line menu on the upper left hand side, and click on the points.

Inquiry 2: HCF & LCM

Two numbers have a HCF of 6. What might the numbers be?

Two numbers have a LCM of 120. What might the numbers be?

Two numbers have a HCF of 6 and a LCM of 120. What might the numbers be?

Find all pairs of numbers whose HCF is 6 and LCM is 120. What do you notice?

Explain the task above or hand out Student Sheet 4 – Finding Numbers Using HCF & LCM.

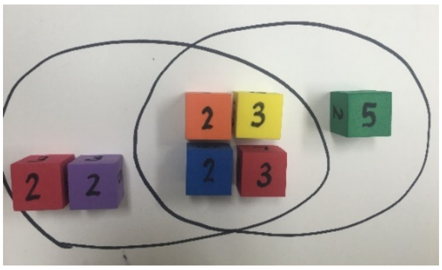
Students might list numbers that have factors of 6 (first question) or list multiples of 120 (second question), and list pairs that have these as common factors or multiples. They should be careful that the pairs chosen do not have higher common factors, such as 12, or lower common multiples, such as 60.

- If one of the numbers is 6, then the other number can be any multiple of 6. If one of the numbers is 12, then the other number can be any multiple of 6 that is not divisible by 4. If one of the numbers is 30, then the other number can be any multiple of 6 that does not have a factor of 5. Many other examples.
- There are 31 pairs of numbers that have a LCM of 120. Some pairs are {120, 120}, {60, 8}, {40, 3}, {40, 6}.

Remind students that they have been exploring prime factorisation and that it might be an efficient way to find HCF and LCM. The factor dice can be used as tool to explore the similarities of the prime factorisations for different numbers.

The pictures show prime dice from Lesson 2 in this sequence being used to find pairs of numbers that have a HCF of 6 and a LCM of 120.

30 and 24 have a HCF of 6 and a LCM of 120.



The HCF is seen (twice) in the middle of the diagram. (Note that to make a Venn diagram of factors, the middle factors would only appear once.) Multiplying the HCF by the numbers in the outer segments gives the LCM which is 120.

The only pairs of numbers that have a HCF of 6 and a LCM of 120 are 6 and 120, and 24 and 30. The product of each of these pairs of numbers is 720.

HCF and LCM Reflection

Pose the questions below verbally or hand out Student Sheet 5 – HCF & LCM Reflection. Give students time to consider each problem and discuss responses and reasons thoroughly.

1. *Explain why we never ask for the lowest common factor or the highest common multiple of two numbers.*

The lowest common factor of any two numbers is always 1. Common multiples can be as large as we like.

2. *Can you find two numbers whose LCM equals the product of the two numbers?*

These numbers will not share any of the same prime factors.

Can you find any two numbers whose LCM is smaller than the product of the two numbers?

These numbers will share a common factor.

Can you find any two numbers whose LCM is larger than the product of the two numbers?

No – the product is itself a common multiple of both numbers, so the least common multiple cannot be larger.

3. *Explain how you can use prime factorisations to find the HCF and LCM of two numbers. Use prime factorisation to find the HCF and LCM of 24 and 60.*

Find the longest factor string that is common to both numbers. For 24 and 60, 2 x 2 x 3 is the longest. This is the HCF.

24 = 2 x 2 x 2 x 3   60 = 2 x 2 x 3 x 5

The HCF of 24 and 60 is 2 x 2 x 3 which is 12.

The LCM is the shortest factor string that contains the prime factorisation of both numbers.
2 x 2 x 3 appears in both factor strings – it is the HCF. We only need to use this string once.

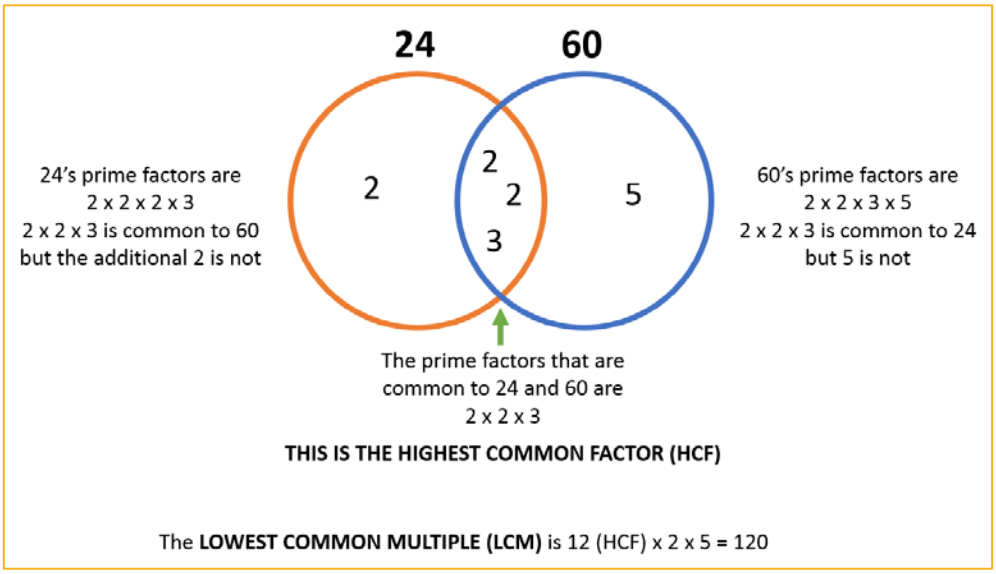
24 = 2 x 2 x 2 x 3   60 = 2 x 2 x 3 x 5

The LCM will not contain any unnecessary factors. This means that the shortest string will be:

2 x 2 x 2 x 3 x 5 = 120.

The LCM is 120.

These ideas can be visualised by showing the two sets of prime factors in an intersecting Venn diagram.



4. *Why is the HCF multiplied by the LCM always equal to the product of the two numbers?*

Why does the HCF (a,b) x LCM (a,b) = ab?

Looking at 24 and 60:

24 = 2 x (2 x 2 x 3) = 2 x HCF

60 = (2 x 2 x 3) x 5 = 5 x HCF

HCF = 2 x 2 x 3

LCM = 120 = 2 x (2 x 2 x 3) x 5

24 x 60 = [2 x (2 x 2 x 3)] x [(2 x 2 x 3) x 5]

= (2 x HCF) x (HCF x 5)

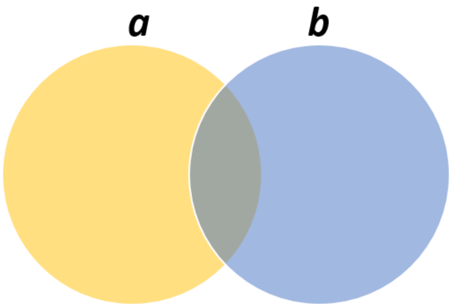
= 2 x HCF^2 x 5

HCF x LCM = (2 x 2 x 3) x [2 x (2 x 2 x 3) x 5]

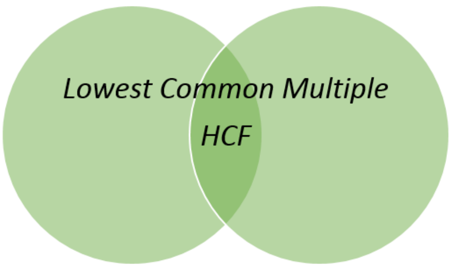
= HCF x 2 x HCF x 5

= HCF^2 x 2 x 5

Using a Venn Diagram:



a x *b* multiples all the factors of *a* and all the factors of *b*



LCM = two complete circles (including the intersection)
HCF = factors in the intersection of the circles
LCM x HCF = the factors in both complete circles by the factors in the intersection of the circles

In both cases the product includes the factors in the intersection twice each.

Additional Tasks and Resources

- The Maths 300 lesson 181 Natalie's LCM (subscription) could be used to explore lowest common multiple. It is a similar problem, and includes a computer investigation of pairs of numbers with a given LCM.
- The app 'Wuzzit Trouble' provides great consolidation of common factors and multiples.
- The following could be used as an extension task:

  Make all the different 3-digit numbers you can using the digits 4, 6 and 8 once each. Find the HCF and LCM of this whole set of numbers.

  Solution:

  There are six 3-digit numbers: 468, 486, 648, 684, 846 and 864.

  Each is divisible by 18 as each is even and the digits sum to a multiple of 9.

  Dividing each by 18 (or 2 x 3^2) we obtain: 26, 27, 36, 38, 47 and 48.

  The prime factors of these numbers are:

26 = 2 x 13	27 = 3^3	36 = 2^2 X 3^2
38 = 2 x 19	47 = 47	48 = 2^4 x 3

  As these have no common factors, the HCF of the original set of six 3-digit numbers must be 18.

  In the LCM we need the maximum number of occurrences of each prime factor in any of these, that is, 2^4, 3^3, 13, 17 and 47.

  So the LCM is 2 x 3^2 x 2^4 x 3^3 x 13 x 17 x 47 = 2^5 x 3^5 x 13 x 17 x 47 = 80 769 312 re(Solve) MATHS BY INQUIRY

Exploring Stars

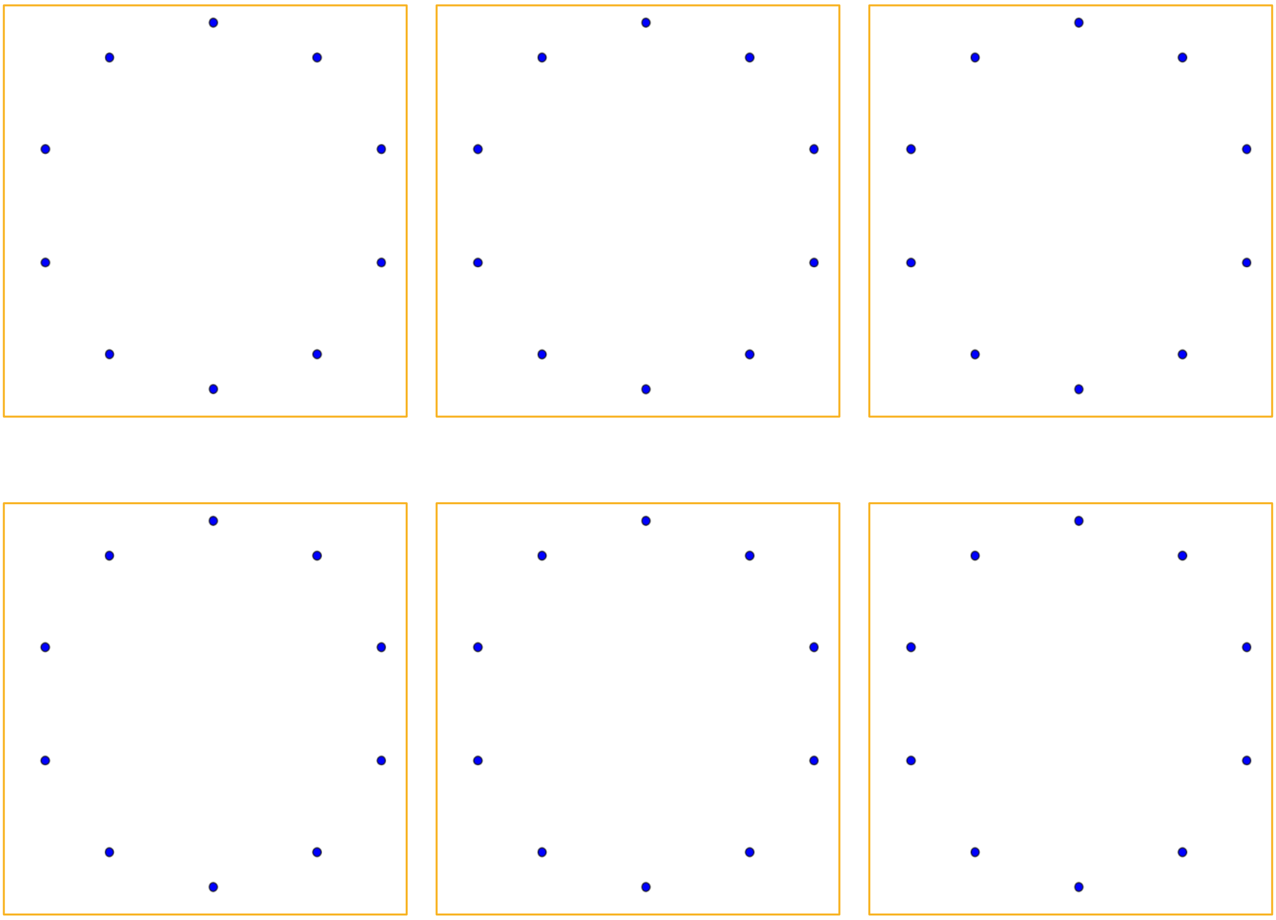
Name: ____________

Here are some sets of 10 points arranged in circles.

Move in a clockwise direction and join every second point with a straight line. Keep going until you get back to where you started. This is a jump size of 2. What shape do you make? What points does the line touch?

Join every third point, which is a jump size of 3. Keep going until you get back to where you started. What shape does it make this time? Does it touch every point? What points does the line touch?

Try some other jump sizes.



Write down what you have discovered about the shapes that have been made.

Make some predictions about the stars that will be made with different sized jumps when the circle has 20 points?

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

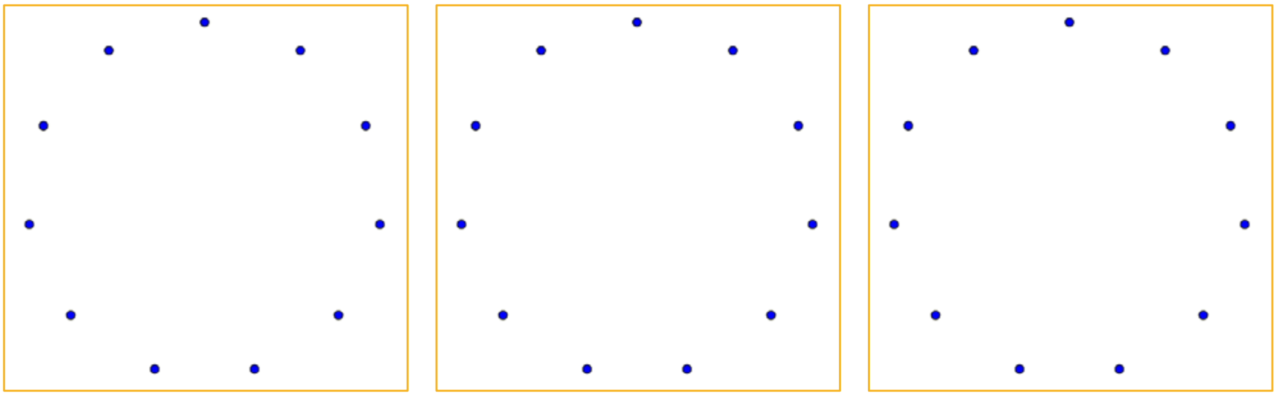
Australian Academy of Science

aamt

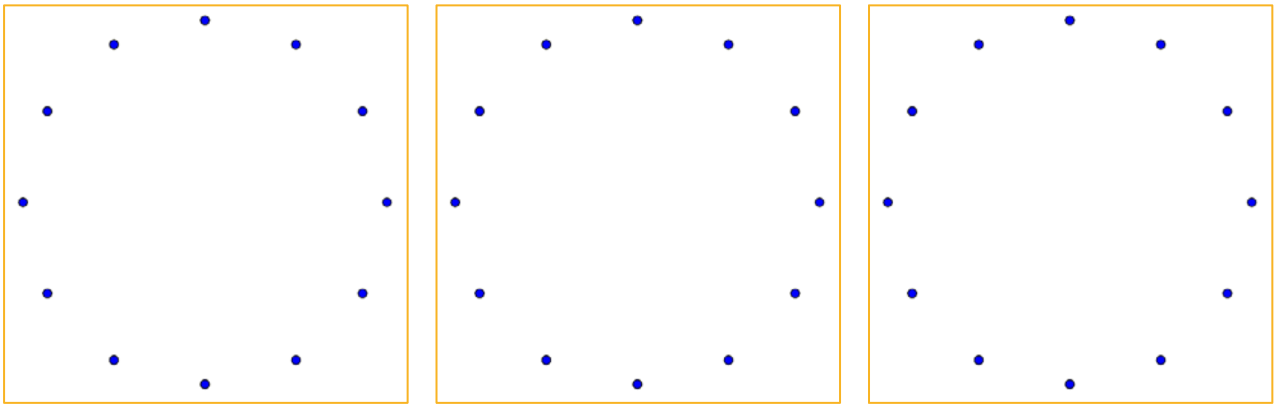
Circle Templates for Drawing Stars

Name: ________________

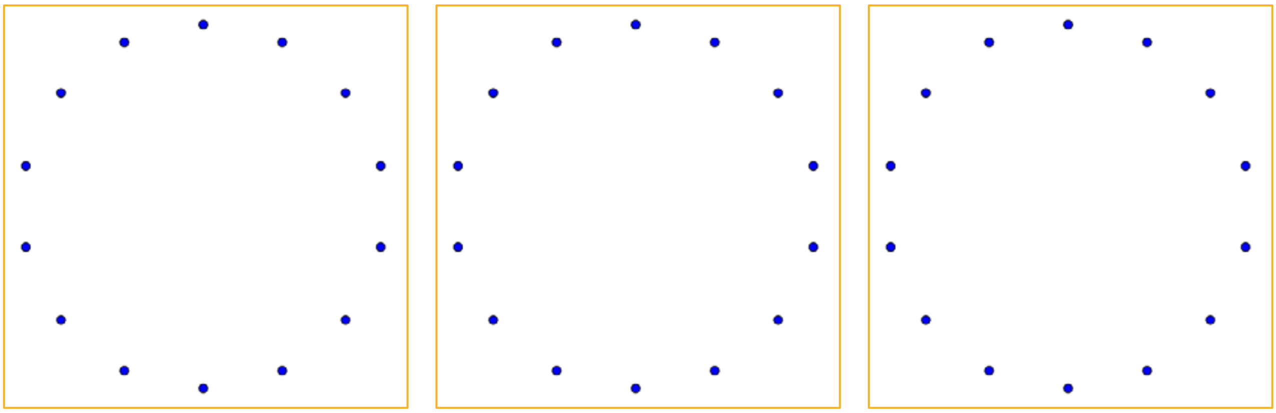
11 point circles



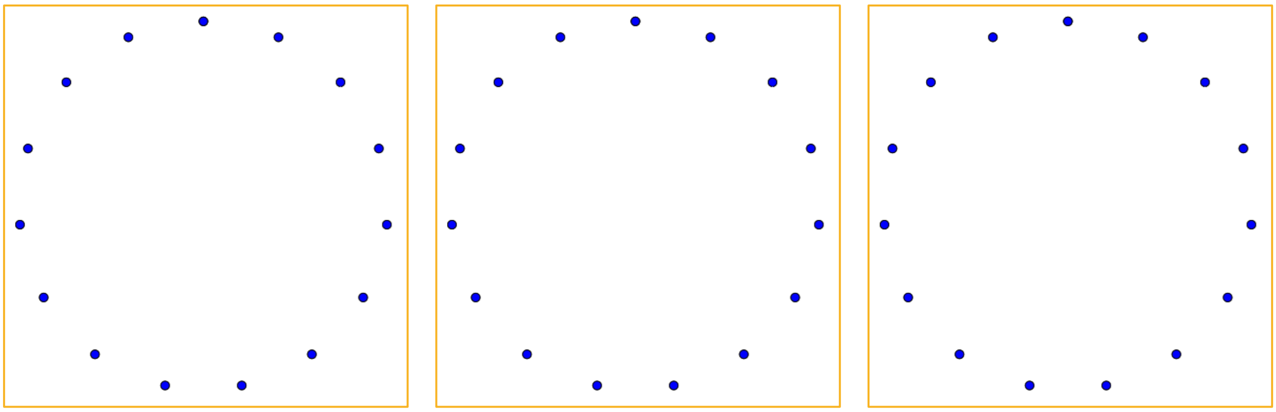
12 point circles



14 point circles



15 point circles



Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science

Exploring Stars Recording Sheet

Name: ______________

Explore different jump sizes around circles with different numbers of points and record your results in the table.

Number of points around the circle	Jump size	Number of points on 'star' shape.	Gap between points on 'star' shape	Number of laps around the circle	Total number of points passed in all laps
10	2	5	2	1	10
10	3	10	1	3	30

Write down some of the patterns that you see in this table.

How can you predict the number of points on the star if you know the number of points around the circle, and the jump size?

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Finding Numbers Using HCF & LCM

Name: ______________

Two numbers have a HCF of 6. What might the numbers be?

Two numbers have a LCM of 120. What might the numbers be?

Two numbers have a HCF of 6 and a LCM of 120. What might the numbers be?

Find all pairs of numbers whose HCF is 6 and LCM is 120. What do you notice?

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science

re(Solve) MATHS BY INQUIRY

HCF & LCM Reflection

Name: ______________

1. Explain why we never ask for the lowest common factor or the highest common multiple of two numbers.

2. Can you find two numbers whose LCM equals the product of the two numbers?

   Can you find any two numbers whose LCM is smaller than the product of the two numbers?

   Can you find any two numbers whose LCM is larger than the product of the two numbers?

3. Explain how you can use prime factorisations to find the HCF and LCM of two numbers. Use prime factorisation to find the HCF and LCM of 24 and 60.

4. Why is the HCF multiplied by the LCM always equal to the product of the two numbers?

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science

aamt