re(Solve) MATHS BY INQUIRY

PRIME FACTORISATION

Lesson 3: Counting Factors

Australian Curriculum: Mathematics – Year 7

ACMNA149: Investigate index notation and represent whole numbers as products of powers of prime numbers.

- Applying knowledge of factors to strategies for expressing whole numbers as products of powers of prime factors, such as repeated division by prime factors or creating factor trees.

Lesson abstract

In this lesson students use the prime factorisation of numbers to explore all their factors. The lesson introduction is a short exploration to identify numbers that have the same number of factors. The major part of the lesson is a substantial investigation that develops a systematic method for generating all the factors of a number given its prime factorisation.

Mathematical purpose (for students)

To discover an easy way to work out how many factors a number has.

Mathematical purpose (for teachers)

Having completed introductory activities relating to prime factorisation in the first two lessons of this sequence, this lesson provides an opportunity for students to apply their knowledge in a substantial investigation. As a result of the activity students learn how to find the number of factors of a number and to generate a complete list of factors from the number's prime factorisation.

Lesson Length Two parts, each 60 minutes approximately.

Vocabulary Encountered

- indices

Lesson Materials

- Paper and post-it notes for class factor display (optional)
- Prime number cards (you will need five large '2' cards, five '3' cards and three cards each of '5' and '7')
- Student Sheet 1 – How Many Factors? (1 per student, optional)
- Student Sheet 2 – Finding All Factors (1 per student)

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Part 1 of the Inquiry: How Many Factors?

Introductory activity

12 has 6 factors. They are 1, 2, 3, 4, 6, and 12.

75, 605 and 847 also have 6 factors.

176 and 405 each have 10 factors.

36 and 441 each have 9 factors.

Can you find another number less than 1000 with 6 factors?

Can you find another number less than 1000 with 10 factors?

Can you find another number less than 1000 with 9 factors?

This introductory activity invites students to gather data leading to the hypothesis that two numbers will have the same number of factors if they have the same number of prime factors and also the same pattern of repeated prime factors. Allow students to explore the problem for a while using their own strategies. Student Sheet 1 – How Many Factors? may be used.

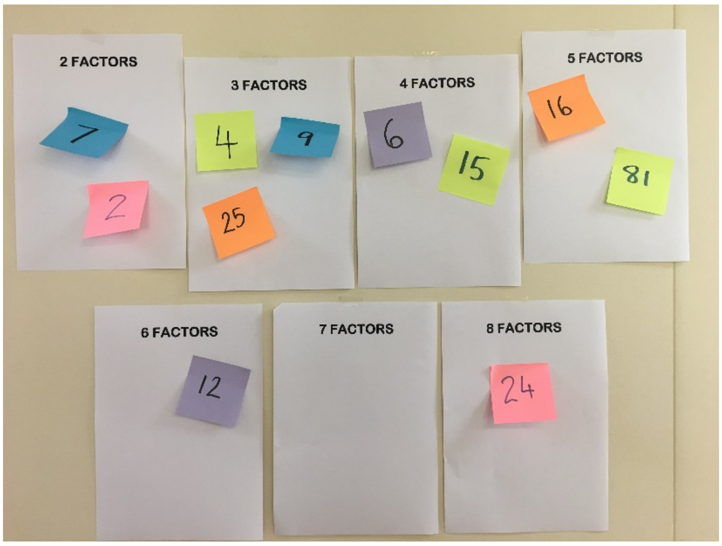
Enabling Prompt

Provide counters and/or grid paper for students with difficulty multiplying. They can investigate what arrays can be made for various numbers. Printed multiplication tables may also be useful.

Encourage some students to use calculators to search for factors or to check their work.

Class factor display

In this section students work together to create a display that organises numbers by how many factors they have.



Students will naturally start with small numbers, listing all the factors. As they find numbers with a given number of factors they can write their numbers on a post-it note and stick them on the class factor display similar to the image shown.

You may wish to assign numbers to groups of students. Encourage students to check the accuracy of the display as it is being created. The more numbers in the display the easier it will be to see the patterns.

Ask students what they notice about the numbers in any given region. For example:

- Primes have exactly two factors.

- Perfect squares have an odd number of factors.
- Numbers with 4 factors are either the product of two primes or the cube of a prime.
- Numbers with 6 factors are divisible by the square of a prime.

Can you see a connection between the form of the prime factorisation and the number of factors?

Add the numbers' prime factorisations to the class factor display. (You might use new post-it notes.)

Students may observe that numbers with a given number of factors have similar prime factorisation patterns and hypothesise that if we know the prime factorisation we can find the number of factors.

They can test their hypothesis by finding other numbers with a similar prime factorisation and listing their factors.

Teacher Notes

- At this stage there is no explanation of why the prime factorisation determines the number of factors – a more formal development of these ideas is in Part 2 of the lesson.

Extending Prompt

- What is the largest number less than 1000 with 6, 10 or 9 factors?
    - A number with 6 factors is either of the form a^5 or a^2b. The fifth powers are not good choices, because $3^5 = 243$ and $5^5 > 1000$. In fact the largest number is $981 = 3^2 \times 109$, with factors 1, 3, 9, 109, 327, 981.
    - The largest numbers with 10 and 9 factors are $976 = 2^4 \times 61$ and $556 = 2^2 \times 13^2$ respectively.

Part 2 of the Inquiry: Finding All Factors

A number has factors that include 1, 2, 8, 12 and 20. What other factors must it have?

What if 17 was also a factor? How many factors would this new number have?

How could we find all the factors of a number from its prime factorisation? You might like to use the number 120 as an example to help you explain.

Allow the students to explore the questions above (also provided on optional Student Sheet 2 – Finding All Factors) for a while using their own strategies.

Some possible reasoning that can be discussed with students:

- The information about factors 1 and 2 is redundant since if it has a factor of 8 it has factors 1, 2, 4 and 8.
- Since it has a factor of 12, it has a factor of 3. So it also has factors 2 x 3, 4 x 3 and 8 x 3.
- It also has a factor of 20, so has a factor of 5. It therefore has factors 2 x 5, 4 x 5 and 8 x 5.
- We can also have factors that include 3 x 5. So we also have 2 x 3 x 5, 4 x 3 x 5 and 8 x 3 x 5.
- In fact, the smallest number with factors 1, 2, 8, 12 and 20 is 2 x 2 x 2 x 3 x 5, or 120.
- It has 16 factors. They are 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60 and 120. We know that we have them all because they come in pairs (1, 120), (2, 120) up to (10, 12). 11 is not a factor of 120, so these pairs are all possible factors.
- If 17 is also a factor, we double the number of factors, because each of these is still a factor, as is each of them multiplied by 17.

Class demonstration

A class demonstration can be used to illustrate how all factors for a given number can be found using its prime factorisation. This example uses cards with the prime factors of 120, but others using larger primes could be used.

Ask students to hold the large prime number cards to represent the prime factorisation for 120. Record the factor pairs as they are calculated.

1 and 120 – Show this with students holding factor cards all standing together → 2 x 2 x 2 x 3 x 5

> Ask the students: *Can we make factors that include only 2s? If so, what are they and what are their pairs?*
>
> To show the factors distinctly one student holding a 2 can stand separate to the other factors, then two students each with a 2, then three students. So the pairs are:

2 and 60 – 2 x (2 x 2 x 3 x 5)

4 and 30 – (2 x 2) x (2 x 3 x 5)

8 and 15 – (2 x 2 x 2) x (3 x 5)

> Ask the students: *Can we make factors that include only 3s? If so, what are they and what are their pairs?*

3 and 40 - 3 x (2 x 2 x 2 x 5) is the only possibility.

> Ask the students: *Can we make factors that include only 5s? If so, what are they and what are their pairs?*

5 and 24 - 5 x (2 x 2 x 2 x 3) is the only possibility.

> Ask the students: *Can we make factors that include 2s and 3s in one factor and 2s and 5s in the other? If so, what are they and what are their pairs?*

6 and 20 – (2 x 3) x (2 x 2 x 5)

12 and 10 – (2 x 2 x 3) x (2 x 5) are the two possibilities.

We now have every possible combination of prime factors. This shows that 120 has 16 distinct factors.

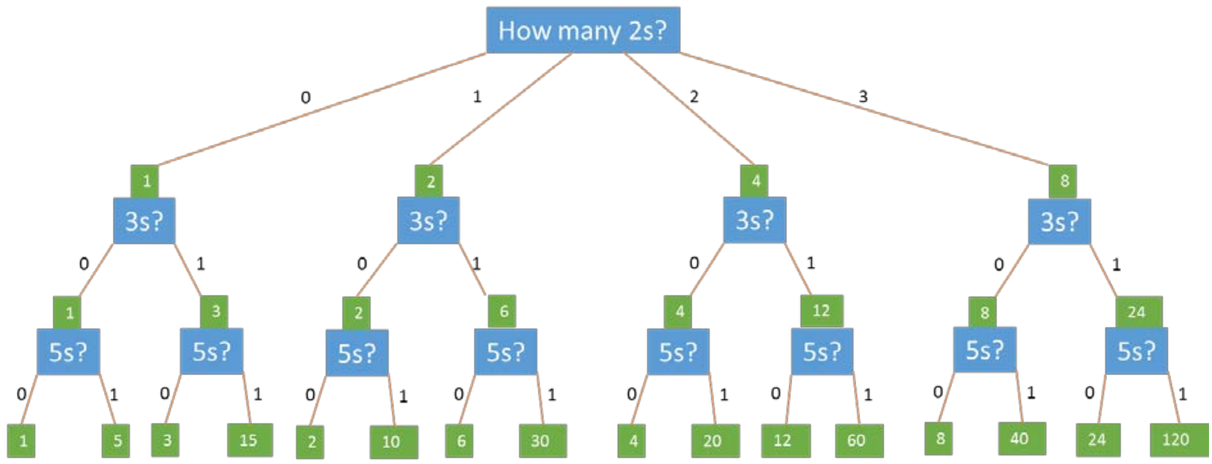
Assessment and reflection

Share students' explanations of how the prime factorisation of a number can be used to systematically determine all of its factors.

Teacher Notes

The activity can be represented using a tree diagram showing the choices made for the number of 2s, 3s and 5s in each factor.

$120 = 2^3 \times 3 \times 5$



This shows that:

- There are 4 pathways for 2 (2^0, 2^1, 2^2 or 2^3) - note that 4 is one more than the exponent (3)
- There are 2 pathways for 3 (3^0 or 3^1) - note that 2 is one more than the exponent (1)
- There are 2 pathways for 5 (5^0 or 5^1) - note that 2 is one more than the exponent (1)

So the total number of factors of 120 is $4 \times 2 \times 2 = (3 + 1) \times (1 + 1) \times (1 + 1) = 16$.

Consolidating Task

Ramanujan's Highly Composite Numbers

Srinivasa Ramanujan was a renowned Indian mathematician. The film *The Man Who Knew Infinity* is based on his life. In 1915, Ramanujan coined the phrase *highly composite numbers*. A highly composite number is a positive number with more factors than any other smaller positive number has.

1 has one factor, which is more than any previous number and so it is highly composite

2 has two factors, which is more than any previous number and so it is highly composite

3 also has two factors, so is not highly composite

4 has three factors, which is more than any previous number and so it is highly composite

5 has two factors, so is not highly composite

6 has four factors, which is more than any previous number and so it is highly composite

What do you think might be the next highly composite number? What is its prime factorisation?

A list of highly composite numbers can be found at https://en.wikipedia.org/wiki/Highly_composite_number

How Many Factors?

Name: ______________

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Finding All Factors

Name: ______________

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