MATHEMATICAL MODELLING

Teachers’ Guide

The missing bridge between school mathematics and the outside world is mathematical modelling.

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Mathematical modelling is the process of using mathematics to make predictions about the real-world, to understand situations and to assist in making decisions. The reSolve special topic *Mathematical Modelling* aims to demonstrate the processes of mathematical modelling, so that students can better use the mathematics they learn to solve problems that arise at home or at work.

This guide outlines the units and provides background information on mathematical modelling and its teaching.

There are five units, especially suitable for Years 9 and 10. After the introduction (Unit 1), the other units can be done in any order. Two or three units might be used per year. The mathematical content involved is mainly drawn from Years 8 and 9, so that students can concentrate on the modelling process. Every lesson is accompanied by a slideshow, and supporting mathematical files (e.g. spreadsheets, dynamic geometry) where necessary.

Outline of the Units

There are five units, each presented as 5 lessons, with accompanying slideshows. All students should begin with introductory Unit 1, but a substitute 'quick start' lesson is supplied in case this is not possible. The quick start lesson gives students something of the flavour of modelling in a single lesson. It could also be used as revision if there has been a long gap since students did Unit 1.

After Unit 1, units can be studied in any order. The models created in the units all start out simple, and can be refined using whatever mathematical skills the students can confidently use. As a guide to unit choice, the geometry in Unit 3 *Packaging Designer* is simpler than the geometry in Unit 4 *Cornering*; and Unit 2 *Pricing for Profit* can be approached with very simple mathematics. In tackling non-routine tasks, students can generally only use concepts and techniques that they have thoroughly absorbed, so the level of mathematics required is deliberately kept low.

If possible, avoid using the units in close proximity to the teaching of the relevant mathematical content, to avoid students simply seeing them as illustrative applications, not modelling tasks. Working out what tools to use is at the heart of modelling - indeed at the heart of all problem solving.

Unit	Description
Unit 1 Introduction to Mathematical Modelling	This introductory unit is designed to help students acquire a deeper understanding of mathematical modelling and the key processes involved. Through modelling two familiar queuing situations – traffic jams and waiting in line at a theme park – students develop an overview of the process of modelling, which is captured in the mathematical modelling cycle diagram. (more details)
Quick Start lesson: The Modelling Process. (substitute for Unit 1)	This lesson introduces the process of mathematical modelling. It emphasises the aspects of formulating, interpreting and evaluating a model as students develop rules for estimating how long a bushwalk will take and later how the number of descendants of someone might grow over a number of generations. (more details)
Unit 2 Pricing for Profit	This unit helps students develop a model of the trade-off in setting prices for a product, in particular the way demand falls with price and how that affects profits. It uses the scenario of students making wooden animals and biscuits to sell for charity at a school fair. Through the lessons, students increase their understanding of the mathematical modelling process. (more details)
Unit 3 Packaging Designer	Students use mathematics to design a container to hold five nearly cylindrical products (e.g. candles, bottles) with exact dimensions not known. They consider the factors important in the design of packaging, including the priorities of the manufacturer (e.g. easily manufactured/engineered and transported), the marketer and retailer (e.g. attractiveness) and of customers. (more details)
Unit 4 Cornering	This unit is about the way in which long vehicles such as buses turn corners, and the consequent road safety aspects of riding, driving or even waiting at a corner alongside a large turning vehicle. The main insight is that the rear wheels of a long vehicle do not follow the path of the front wheels, but 'cut the corner'. This has implications for parking, road widths etc. Students start with paper scale models, then examine the path of the wheels of a bike and use pre-made dynamic geometry programs that highlight the geometric features of cornering considering the vehicles' width and front and back overhangs. (more details)
Unit 5 How Risky is Life?	Students develop insight into the mismatch between real and perceived risk – something that affects and sometimes impoverishes people's lives. They explore Australian Bureau of Statistics data on the risks of dying unexpectedly from various causes, coming to recognise that some risks are exaggerated by the media. They learn how risks vary with age and gender. The emphasis is on order-of-magnitude comparisons of small numbers. (more details)

What is mathematical modelling?

Why teach modelling?

There are compelling reasons to include mathematical modelling in the school curriculum, including these.

- Many students leave school without enthusiasm for mathematics. Modelling, because it takes real world contexts seriously, helps motivation for many.
- Though most students know that mathematics is important for academic and career progress, they don't see the mathematics they are taught as useful for, or relevant to, their lives beyond school, except for simple calculations. They have not thought about how mathematics is really used.
- Many adults underestimate the mathematical thinking that is involved in their work, because they only think of mathematics as the topics (such as quadratic equations) learned at school. In fact, many jobs involve more generic mathematical thinking – organizing, making estimates, devising plans, using spreadsheets, for example.

The missing bridge between school mathematics and the outside world is mathematical modelling.

Everyone uses mathematical models

Mathematical modelling, the use of mathematics to give greater insight into practical problem situations, is a familiar activity and the outcomes of mathematical modelling are now often discussed in the news. Some of the high profile mathematical models are incredibly complex, such as those used to predict the weather, or the effect of increased carbon dioxide in the atmosphere, or to predict the effect of a cut in taxation on small business activity. Others are simple, such as the instructions on medicine to calculate doses for children, or to find the cooking time for a roast.

At the simplest level, everyone has been doing mathematical modelling for most of their lives: counting the plates to decide if there are enough on the table, working out whether you will have enough cash to pay for your basket of shopping at the supermarket, and deciding which bank account is best for you. These are elementary examples, but they provide a foundation on which to build student confidence as they develop their ability to use mathematics in analysing more complex situations.

What fewer students have experienced is applying this kind of thinking in their mathematics classroom, where most time is spent in learning mathematical procedures and concepts. Changing their expectations in this regard is at the heart of the *reSolve* project as a whole, where students learn to tackle problems for which they have not been taught a specific solution method, drawing on their toolkit of mathematical concepts and techniques to find a method, and then to carry it through. Many of these non-routine problems in *reSolve* lessons are set in practical contexts but often the focus is on a specific area of mathematics – they are *illustrative applications*. In contrast, in mathematical modelling the focus is on the practical situation, with mathematics as a set of tools for developing deeper functional understanding of the situation – the prime criterion of success. The diagram in Figure 1 below illustrates the distinction.

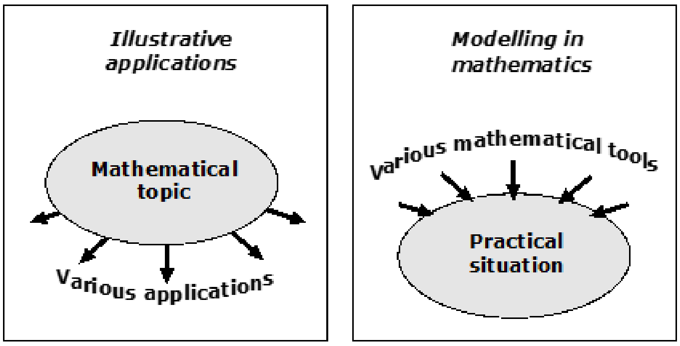


Figure 1. Models v Modelling

What is a mathematical model?

A mathematical model is an abstract mathematical representation of how factors in a situation in the real world are structurally connected. It provides insight into the situation and can be used to answer certain questions that arise.

For example, consider planning how much time to allow for a bushwalk, the problem considered in the 'quick start' lesson. The distance and therefore time a bushwalk takes varies. Other factors such as the number of rests and the age of the walkers will also mean the time walked will vary.

However, bushwalkers often use a rule of thumb called Naismith's Rule to estimate the time to allow. This says that a walk takes 1 hour for each 5 km plus 1 hour for each 600 m ascended. This is a simple mathematical model that is based on two important factors whilst ignoring factors that it could be taken into account but which it would be difficult to do - such as the different terrain, extreme heat or wind, and the fitness of the walkers.

Naismith's Rule is a mathematical model for the time of a bushwalk, even when it is expressed in words. If required, it could be expressed in mathematical notation as $T = \frac{D}{5} + \frac{H}{600}$, where T is the estimated time in hours, D is the distance of the walk, and H is the number of metres ascended.

Obviously, Naismith's Rule is not an exact prediction. However, success for a mathematical model is not precision, but whether it gives answers that are good enough for the practical purpose. It also has the advantage of being easy to remember, and easy to calculate, so suitable to use when on the move.

Naismith's Rule, like all mathematical models, has several components:

- Selection of the *most important variables* to include in the model (in this rule, only distance and height)
- A set of *assumptions*, preferably built on sound knowledge of the situation (in this rule, that a typical walking speed is around 5 km/hour, and that an ascent of 600m slows the walkers by an hour)
- A set of *relationships* (e.g., the distance – speed – time relationship).

It is important that the representation that constitutes the mathematical model captures the structural relationship between the different factors that it takes into account and that it *can be used again and again* for different scenarios rather than just solve a particular problem. These important factors distinguish models from other mathematical representations.

The criterion for successful modelling is to develop a formula (or other well-defined method) that provides useful answers to a class of practical problems. As such, it is the fit of the mathematics with the situation that is most valued, rather than the complexity or ingenuity of the mathematical techniques used. Students are already familiar with this principle in other subjects. When working on essays in English, for example, the techniques of writing, grammar, punctuation and so on are important but they are judged in relation to the quality of the reasoning about the situation.

The modelling process and the modelling cycle diagram

There is a strong reflective and meta-cognitive element in the units. The units use a diagram of the process of mathematical modelling, shown in Figure 2 below. This particular diagram of the modelling process has been chosen because of its simplicity, and the way it highlights the movement between real-world and mathematical thinking.

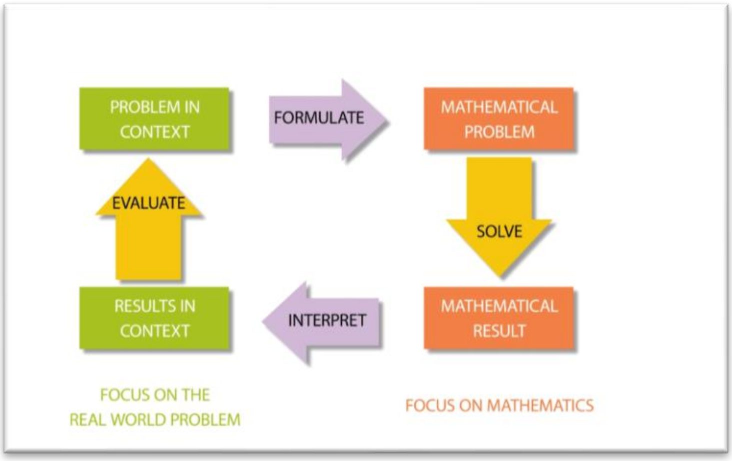


Figure 2. The modelling cycle

The main features of the diagram are as follows.

- Modelling begins with a problem in context. The first step is to understand the context and the problem well.
- The purple arrows move between the real world understanding of the problem and its mathematical representation.
- A mathematical problem statement is formulated from this real problem, moving in the diagram from the 'green' side of the real world to the 'orange side' of the mathematical world.
- The 'mathematical problem' in an orange box of Figure 2 is the start of the model (perhaps after some reorganisation). Relevant quantities that are significant in the context can become variables. The relationships between them may become formulas, functions or equations to build into the model.

- Within the mathematical world (orange boxes), mathematical activity ‘solves’ the mathematical problem and produces an answer in mathematical terms.
- The purple interpret arrow brings the mathematical work back into the context of the real-world problem, for interpretation - what does this mathematical answer mean in practice. There is a change from mathematical language to real world language.
- The model also needs evaluation; judging whether the solution provides an adequate and sensible answer to the real-world context. If yes, the model can now be used. If no, a new model can be formulated, beginning by thinking more about the real situation, and perhaps making different assumptions, including previously overlooked variables or by re-thinking the relationships between the variables.

Unit 1 and the ‘quick start’ lesson provide detailed information about these phases, illustrated in simple modelling tasks.

Key modelling features in the units

- *Familiar Situations and Familiar Mathematics*
  The situations for modelling in these units were chosen to be familiar to students though it is unlikely they have thought about any of them analytically.
  The mathematics that is essential for the units is intended to be very familiar to them. Students need to be able to focus on the modelling, and not have to concentrate on the calculations. There is a ‘few year gap’ between the mathematics students can reproduce in routine exercises and what they can deploy in tackling non-routine problems. There are many opportunities to extend the level of mathematics when desired.

- *Understanding the Problem in Context*
  Discussion in small groups and with the whole class is recommended to help students understand the real-world situation in some depth. The factors that emerge as important from this process will be built into the model. Rarely can a situation be modelled in a way that takes account of all reasonable factors. Usually, the modelling will need to focus on some aspects of the situation and ignore others, at least initially. Keep a list of the factors that have been ignored for later use.

- *Start Simple*
  Students are encouraged in all units to start with a simple model. Starting simple has two advantages: it is easier to formulate a model, and doing that leads you to think about the situation more deeply. This should be the beginning of the cycle of improvement that modelling diagram shows.

- *Give Students a Consultant Role*
  The units generally put the student in the role of a consultant for a client. For example, in Unit 1, the student is a consultant to the theme park manager. Asking students to solve a problem for someone else has two advantages. Consultants have a certain prestige and they are expected to explore the problem thoroughly. Consultants also need to report to their client, usually in writing.

- *Allocate Time to Prepare a Report*
  The units encourage students to report to the client. Some reports can be verbal, some can be brief posters, sometimes reports should be carefully argued written work. Reports should communicate the insights found, and recommend a course of action. Especially in a classroom, the modeller needs to explain the reasoning that led to these conclusions and recommendations, not in every detail but enough to see the roles of the various factors that were and were not taken into account.
  The creation of a plausible explanation of a chain of reasoning is a powerful form of metacognition - reflecting on the reasoning and the thinking processes you went through, then boiling them down to a clear explanation is a powerful element in long-term learning.

- *Numerical and Algebraic Methods*
  Students are encouraged to work numerically and then algebraically. Most serious modelling usually needs numerical methods; it is rare to find complex realistic situations where algebraic functions provide an adequate model. Fortunately, information technology now makes numerical methods little more laborious than analytic methods. Algebra remains valuable for providing a clear way to express relationships between variables, particularly when the are several variables involved. Algebra also provides the language for the formulas in spreadsheets, albeit a slightly different dialect.

The pedagogy of teaching modelling

Teaching mathematical modelling can present a teacher with many classroom challenges. This section outlines how these challenges may be effectively addressed. (Further resources are also provided.)

Changing the Classroom Contract

In every classroom there is a "classroom contract" (Brousseau 1992), usually implicit, that embodies the shared expectations of the students and the teacher as to what each will do, and expect of the other.

Take, for example, the traditional “3X” mathematics teaching, the teacher eXplains a new procedure, illustrates it by working through an eXample before asking the students to tackle a set of closely similar tasks as eXercises. The student expects to listen (more or less attentively) to the explanation, then to focus more intently on the worked example because it models the procedure they will then try to apply to the set of tasks. This contract clearly does not cover tasks that are significantly different from anything the students have been shown how to solve.

When we introduce non-routine problems, the teacher is faced with two new challenges:

- How to support students without changing the task – for example, by breaking it up into a sequence of tasks, but not too much;
- How to establish a productive knowledge-building classroom culture.

These units have been developed to enable teachers to address these challenges effectively. They embody, and make explicit, the changed expectations that learning to model with mathematics requires and provide detailed suggestions for teachers on moves that will forward that learning process. Of course, every teacher does things their own way in their own classroom but experience (and a lot of feedback) have shown that it is usually helpful to see moves that have worked well for other teachers.

Ideas for changing the classroom contract are best, and most enjoyably, developed through discussion among teachers before and after shared classroom experiences – a “Sandwich Model". Materials to support such DIY in-school professional development sessions on these issues can be found at http://www.bowlandmaths.org.uk/pd/

Promoting reasoning and explanations

Here are some ideas approaches that encourage explanation and reasoning in class.

Teacher’s role in promoting reasoning and explanations	
Invite students to elaborate	Can you just say a little more about that ... What would happen if ...? Can someone suggest another way of doing this? Would anyone like to ask Zac a question about that?
Emphasise reasoning	Can you explain why that works? Can you explain that again? Please take us all through that step by step. Let's think this through together ...
Ask questions in ways that include everyone	Use a 'no hands up' rule and instead call on anyone Ask questions that encourage a range of responses Avoid teacher-student-teacher-student ‘ping-pong’
Make sure that students know that they must take time to think before responding	Try out the answer on your partner first. Use Think - Pair - Share. Get students to jot down ideas on mini-whiteboards

Supporting students when they are tackling unstructured problems

The modelling units help students to learn to use their mathematics to make sense of situations that arise outside of the classroom. For this they need to work autonomously on unstructured problems. This presents a challenge to teachers: *How should I help without taking over?*

There is a lot of pressure on teachers to tell students what to do. However, if students are to learn to solve problems, they have to learn how to decide *what* to do and *when* to do it. If someone always tells them what to do, they *won't* learn these skills for themselves. So, aim to provide less and less guidance as you get further into this kind of work.

Aim to use questions that make students think about the way they are tackling the task, and that highlight general principles. Support can be mostly in the form of questions, not explanations or instructions.

Principles for supporting students	*Sample intervention*
Allow students time to understand and engage with the problem Discourage students from rushing in too quickly or from asking you to help too soon.	*Take your time, don't rush in.* *What do you know?* *What are you trying to do?* *What is fixed? What can be changed?* *Don't ask for help too quickly - try to think it out together.*
Usually offer strategic rather than technical hints Avoid simplifying problems for students by breaking it down into very small steps. Avoid hints referring to specific mathematical moves or results.	*How could you get started on this problem?* *What have you tried so far?* *Can you try a specific example/a simple case?* *How can you be systematic here?* *Can you think of a helpful representation?* *How can we organise this? Let's draw up a table of results.* *Have you checked if that works?* *(NOT: Do you recognise square numbers? or What did we learn yesterday that might be useful?)*
Frequently use questions that make students think about the way they are tackling the task (metacognition)	*What have you tried?* *What options have you thought of?* *What are you really trying to do here?* *What will you do with this result when you get it?* *What have you found out so far?* *Have you seen anything that is like this in any way?*
Encourage students to consider alternative methods and approaches	*Is there another way of doing this?* *Describe your method to the rest of the group* *Which of these two methods do you prefer and why?* *Compare your method with someone from another group*
Encourage explanation Make students do the reasoning and encourage them to explain it to one another.	*Can you explain your method?* *Can you explain that again differently?* *Can you put what Sarah just said into your own words?* *Now write that down as clearly as you can.*
Model thinking and powerful methods When students have done all they can, they will learn from being shown a powerful, elegant approach.	*Now I'm going to try this problem myself, thinking aloud.* *I might make some mistakes here - try to spot them for me.* (Do not do this at the beginning. Students will imitate the method and not appreciate why it was needed.)

Helpful classroom talk

There is overwhelming evidence that *mathematical discussion*, when students engage with each others' reasoning, is beneficial for learning. But what types of talk engage students, develop understanding and promote deeper thinking?

Robin Alexander (2017) identified the following five principles of helpful classroom talk - which he terms *dialogic*.

- *Collective*
  teachers and students address learning tasks together, as a group or as a class, rather than in isolation

- *Reciprocal*
  teachers and students listen to each other, share ideas and consider alternative viewpoints

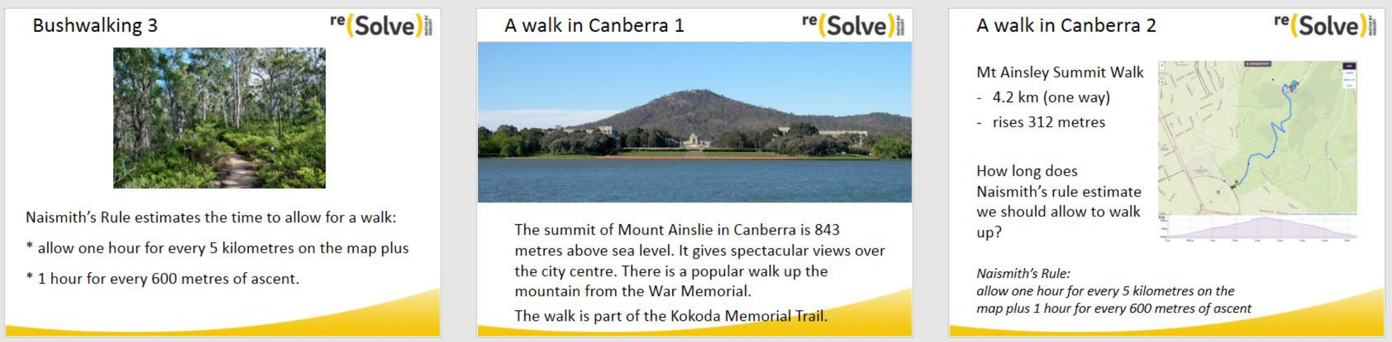
- *Cumulative*
  teachers and students build on their own and each other's ideas and chain them into coherent lines of thinking and enquiry

- *Supportive*
  students articulate their ideas freely, without fear of embarrassment over 'wrong' answers and they help each other to reach common understandings

- *Purposeful*
  teachers plan and facilitate dialogic teaching with particular educational goals in view.

Promoting productive whole class discussion

Teacher's Role in Fostering Productive Classroom Discussion	
Mainly be a Chairperson or Facilitator	Direct the flow of the discussion and give everyone a chance to participate Do not interrupt or allow others to interrupt the speaker Value everyone's opinion Help students to clarify their own ideas in their own terms
Occasionally be a Questioner or Provoker	Introduce a new idea when the discussion is flagging Follow up a point of view Play devil's advocate Focus in on an important concept Avoid asking questions that only require monosyllabic answers
Rarely be a Judge or Evaluator	Support with non-verbal interest Avoid responding with closed, evaluative comments (e.g. yes, no) Try not to sum up prematurely.



Images from Quick Start Lesson

Promoting productive small group discussion

In addition to whole class discussion, collaborative work in small groups is a key feature of knowledge-building classrooms. This section addresses what the teacher can do to make this discussion an effective contributor to students' learning. When students work together, sometimes they build positively, but uncritically on what others have said. They repeat and confirm, but do not really think about the contribution. At other times, there are short exchanges consisting of assertions and counter-assertions, sometimes ignoring others, and sometimes disagreeing.

The type of talk that is most effective for learning is when speakers work on and elaborate each other's reasoning in a collaborative, rather than competitive atmosphere. The reasoning to become audible and knowledge becomes publicly accountable. It is characterised by critical and constructive exchanges. Challenges are justified, and alternative ideas are offered (Mercer, 2012).

First, there are some suggestions for planning for good classroom discussion. Then some suggestions for establishing ground rules for student behaviour in small groups are given. Finally, the teacher's role during small group discussions is canvassed.

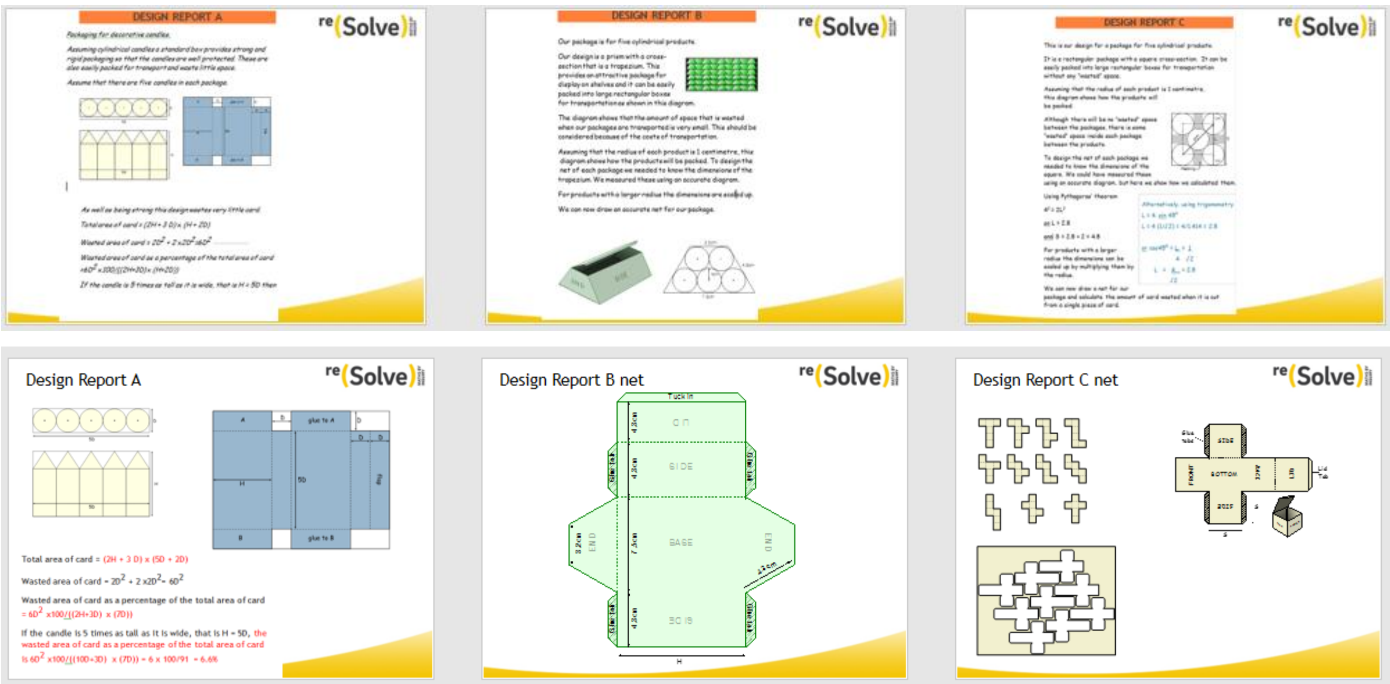
Planning

Planning for small group discussion	
Prepare shared tasks in a form that encourages discussion.	Provide resources *to share* (e.g. one copy between two or four) and ask for outputs that are jointly produced. Provide *big* resources so that reasoning may be visible and shared, such as large sheets of paper, felt-tipped pens or 'mini- whiteboards'. Sometimes require *joint outcomes* where students share responsibility e.g. a poster or a report.
Plan how you will arrange the room	Arrange tables and chairs so students face each other while working together Share computers
Plan how you will group students	Most students are more able to discuss in smaller groups than larger ones: pairs or threes is often most effective. Sometimes use a *snowball* approach: • Students first tackle the task individually. They have time to think before they are asked to discuss. • Pairs are then formed, and students are asked to try and reach agreement. • Pairs then join together so that a broader consensus might be reached. • Groups of four then report back to the whole class in a plenary discussion.
Plan how you will introduce the purpose of discussing	Plan your introduction to pre-empt the questions: • Why do you want us to discuss? • What do you want us to do? *This lesson is not about 'me showing you a method and then you using it'. No, I want to see if you can find your own methods. There is more than one way of doing this! I want you to discuss your own ideas for starting on this problem.*
Plan how you will establish ground rules	Introduce ground rules for students (see below). Such behaviours are not established overnight, but over a long time through consistent reinforcement.
Plan how you will end group discussion	Most teachers ask students to report back on their discussion in some way. Warn all students to prepare for this. Try not to pass judgments on their responses while they do this, or this may influence subsequent contributions. Be sure to follow up on issues that need to be addressed.

Setting Discussion Ground Rules

Here are some 'ground rules' for **students** to use as they work in groups. These could be displayed in the classroom and reviewer and reinforced over time. You might like to draw up such a list with the class.

Discussion ground rules	Sample student behaviour
Give everyone in your group a chance to speak	*Let's take it in turns to say what we think.* *Claire, you haven't said anything yet.*
Listen to what people say	*Don't interrupt - let Sam finish.* *I think Sam means that ....*
Try to understand what is said	*I don't understand. Can you repeat that please?* *Can you show me what you mean?*
Build on what others have said	*I agree with that because ...* *Yes, and I have also noticed that ....*
Demand good explanations	*Why do you say that?* *Go on ... convince me.*
Challenge what is said	*That cannot be right, because...* *This explanation isn't good enough yet.*
Treat opinions and contributions with respect	*That is an interesting point.* *Don't worry, I made that mistake too!*
Share responsibility	*Let's make sure that we are all able to report this back to the whole class.*
Reach agreement	*We've got the general idea, but we need to agree on how we will present it.*



Images from Packaging Designer Lesson 3

Teachers' role during small group discussions

The teacher has an active role whilst students are discussing with their peers. The pressing task is to assist the groups to have a productive discussion. The table below shows how this might be done. The concurrent task is to gather insight into the work of the groups, so that the follow-up teaching (usually some sort of plenary session) is as effective as possible.

Teacher's role during small group discussions	
Make the purpose of the task clear	Explain what the task is and how they should work on it. Also, explain why they should work in this way. *'Don't rush, take your time. The answers are not the focus here. It's the reasons for those answers that are important. You don't have to finish, but you do have to be able to explain something to the rest of the class.'*
Keep reinforcing the 'ground rules'	Try to ensure that students remember the ground rules that were discussed at the beginning. Encourage students to develop a responsibility for each other's understanding. *'I will pick one of you to explain this to the whole class later – so make sure all of you understand it'.*
Listen before intervening	When approaching a group, stand back and listen to the discussion before intervening. It is all too easy to interrupt a group with a predetermined agenda, diverting their attention from the ideas they are discussing. This can be annoying and disruptive for the group and interrupt concentration.
Join in, don't judge	Try to join in as an equal member of the group rather than as an authority figure. When teachers adopt judgmental roles, students tend to try to 'guess what's in the teacher's head' rather than try to think for themselves. *'Do you want us to say what we think, or what we think you want us to say?'*
Ask students to describe, explain and interpret	The purpose of an intervention is generally to increase the depth of reflective thought. Challenge students to describe what they are doing, to interpret something ('*Can you say what that means?*') or to explain something ('*Can you explain the reasoning behind that?)*
Do not do the thinking for students.	Many students are experts at making their teachers do the work! They know that if they 'play dumb' long enough, then the teacher will eventually take over. Try not to fall for this. If a student says that he or she cannot explain something, ask another student in the group to explain, or ask the student to choose some part of the problem that she can explain. Don't let them off the hook! When a student asks the teacher an interesting question, don't answer it (at least straight away). Ask someone else in the group to do so.
Don't be afraid of leaving discussions unresolved (for a while!).	Some teachers like to resolve discussions before they leave the group. When the teacher leads the group to the answer, the discussion has ended. Students are left with nothing to think about. It is often better to reawaken interest with a further interesting question that builds on the discussion and then leave the group to discuss it alone. Return some minutes later to find out what has been decided.

Assessing modelling

Assessment of performances plays a central role in the development of any set of skills and understanding. This section considers the different ways that progress in modelling can be assessed, especially as formative assessment during teaching. A sample assessment rubric based around the modelling processes is given, along with some pointers for involving students in self and peer assessment.

Some principles for assessing modelling processes

These principles reflect the characteristics of formative assessment, which may be defined as:

"... all those activities undertaken by teachers, and by their students in assessing themselves, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged. Such assessment becomes 'formative assessment' when the evidence is actually used to adapt the teaching work to meet the needs." (Black and Wiliam 1998)

Assessment needs to be used to promote learning, and teachers need to focus on the kinds of feedback that are most helpful to promote learning.

- *Make the process objectives explicit.*
  Share the process objectives with students and from time to time ask students to produce evidence that they can achieve these objectives. This is sometimes difficult as students find processes less understandable than content. This doesn't mean writing them on the board at the beginning of the lesson, but rather referring to their use consistently and explicitly while attempting to solve unstructured, non-routine problems. In plenary sessions, ask students to share and compare approaches, rather than answers. When students get stuck, offer strategic advice on what they need to do next.

- *Assess groups as well as individual learners.*
  Group activities such as poster-making allow many opportunities to observe, listen, and question learners. They make thinking visible and allow the teacher to see quickly where difficulties have arisen. Collaborative work needs to be combined with individual responsibility.

- *Use divergent assessment methods ("Show me what you know about ...").*
  Convergent assessment strategies are characterised by tick lists and can-do statements. The teacher asks closed questions to ascertain whether the learner knows, understands or can do a predetermined thing. This is the type of assessment most used in written tests. Divergent assessment, in contrast, involves asking open questions that allow learners opportunities to describe and explain their thinking and reasoning. These questions allow learners to surprise us - the outcome is not predetermined.

- *Give constructive, useful feedback.*
  Research shows that responding to students' work with marks or levels is ineffective and may even obstruct learning. Quantitative feedback of this type results in students comparing marks or levels and detracts from the mathematics itself.
  Instead, use qualitative oral and written comments that help learners recognise what they can do, what they need to be able to do and how they might narrow the gap.

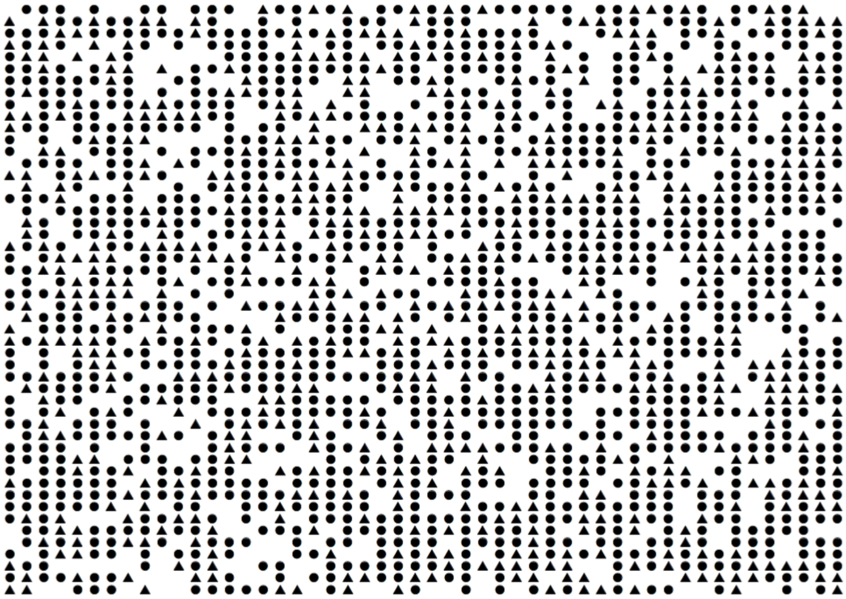
- *Change teaching to take account of assessment findings.*
  As well as providing feedback to learners, good assessment feeds forward into teaching. Be flexible and prepared to change your teaching plans in mid-course as a result of what you discover.

An illustrated assessment rubric for 'Counting Trees'

This section presents a simple sample modelling task, some student responses and an assessment rubric.

The rubric has three columns linked to the modelling cycle, and a fourth for communication. Naturally, the quality of responses in each of the columns is linked, but it is worthwhile trying to give feedback to students on how they perform in each section. The first process, Formulating, is very influential. Students who do not formulate an adequate mathematical model may not be able to show their capacities in other processes to best advantage. This can be used to underline to students the importance of thinking very carefully in the first phase of modelling.

Counting Trees



This diagram shows some trees in a plantation.

The circles ● show old trees and the triangles ▲ show young trees.

Tom wants to know how many trees there are of each type, but says it would take too long counting them all, one-by-one.

1. What method could he use to estimate the number of trees of each type? Explain your method fully.

2. On your worksheet, use your method to estimate the number of:
   (a) Old trees
   (b) Young trees

Naima estimates the number of trees by multiplying the number along each side of the whole diagram. She halves the total, not considering that there may be different number of trees of each kind.

① You could multiply the number of trees in the length by the number of trees in the width and then half your answer.

② a. Old trees - 644
Young trees - 644

width - 33
length - 39.

$33 \times 39 = 1287$
$1287 \div 2 = 643.5 - 644$

1. there are 38 trees in each column there are around 11 young trees and around 27 old ones 33 trees in each row so

$11 \times 33 = 363$
$27 \times 33 = 891$
1254

2.
a. $11 \times 33 = 363$ = new trees.
b. $27 \times 33 = 891$ = old trees.

Utkarsh realises that sampling is needed, but he multiplies the number of young trees and old trees in the left hand column by the number of trees in the bottom row. He ignores the columns with no trees in the bottom row, so this method underestimates the total number of trees. He does, however, take account of the different numbers of old and new trees.

Tsu-Nan uses a sample of two columns and counts the number of old and young trees. He then multiplies by 25 (half of 50 columns) to find an estimate of the total number.

2 columns has 21 young trees
55 old

50 columns is approx
50 ÷ 2 = 25
25 × 21 = amount of young trees = 525
25 × 55 = amount of old trees = 1,375
rounded up
young 530
old 1,380

Amber chooses a representative sample and carries through her work to get a reasonable answer. She correctly uses proportional reasoning. She checks her work as she goes along by counting and recording the gaps in the trees. Her work is clear and easy to follow.

Counting trees

1. If Tom draws a 10X10 square round some trees and counts how many old and new there are. There are 50 rows and 50 columns altogether so he must multiply by 25. He could do this a few times to check and then take the average.

2.
53 old x 25 = 1325 old
28 new x 25 = 700 new
19 spaces x 25 = 475 spaces
100 2500

1325 + 1200 ÷ 2 = 1262·5
700 + 875 ÷ 2 = 787·5

Check
48 old x 25 = 1200 old
35 new x 25 = 875 new
17 spaes x 25 = 425 spaes
100 2500

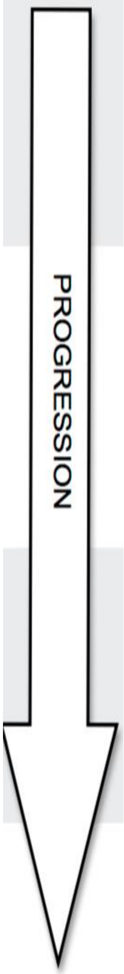
So about 1263 old trees and 788 new Trees

Sample assessment rubric

Formative assessment focuses the student's attention on the many aspects of their modelling reasoning and how it might be improved, in detail and overall. Summative assessment usually requires these many micro-judgments to be condensed into a few levels, often expressed in grades or numbers, thus discarding most of the information in the student's response. This reflects the different needs of different users and purposes. The skilled teacher and the students can make good use of the rich detailed data but school systems, for purposes like student selection or accountability, need summary data.

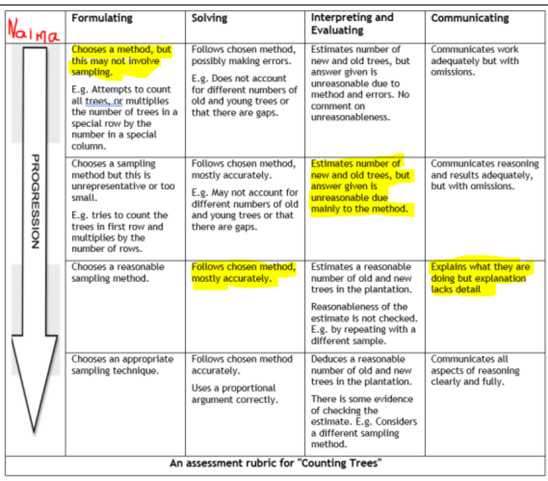
The assessment rubric below can used in various ways for different purposes. In the classroom, it helps the teacher focus on different aspects of the response and guides follow-up questioning. As summative assessment, it can be used to give each response 4 scores or by addition, just one. Because formulating is the key modelling step, this could be weighted more heavily. How to use, and improve, this scheme can stimulate lively discussions with colleagues.

This rubric gives three separate ratings for parts of the modelling cycle, with four levels of performance for each. Interpreting and Evaluating have been combined because in the classroom situation, meaningful evaluation against real world criteria has not been possible. Communication is also assessed.



Formulating	Solving	Interpreting and Evaluating	Communicating
Chooses a method, but this may not involve sampling. E.g. Attempts to count all trees or multiplies the number of trees in a special row by the number in a special column.	Follows chosen method, possibly making errors. E.g. Does not account for different numbers of old and young trees or that there are gaps.	Estimates number of new and old trees, but answer given is unreasonable due to method and errors. No comment on unreasonableness.	Communicates work adequately but with omissions.
Chooses a sampling method but this is unrepresentative or too small. E.g. tries to count the trees in first row and multiplies by the number of rows.	Follows chosen method, mostly accurately. E.g. May not account for different numbers of old and young trees or that there are gaps.	Estimates number of new and old trees, but answer given is unreasonable due mainly to the method.	Communicates reasoning and results adequately, but with omissions.
Chooses a reasonable sampling method.	Follows chosen method, mostly accurately.	Estimates a reasonable number of old and new trees in the plantation. Reasonableness of the estimate is not checked. E.g. by repeating with a different sample.	Explains what they are doing but explanation lacks detail
Chooses an appropriate sampling technique.	Follows chosen method accurately. Uses a proportional argument correctly.	Deduces a reasonable number of old and new trees in the plantation. There is some evidence of checking the estimate. E.g. Considers a different sampling method.	Communicates all aspects of reasoning clearly and fully.

An assessment rubric for "Counting Trees"


Naima

Formulating	Solving	Interpreting and Evaluating	Communicating
Chooses a method, but this may not involve sampling. E.g. Attempts to count all trees, or multiplies the number of trees in a special row by the number in a special column.	Follows chosen method, possibly making errors. E.g. Does not account for different numbers of old and young trees or that there are gaps.	Estimates number of new and old trees, but answer given is unreasonable due to method and errors. No comment on unreasonableness.	Communicates work adequately but with omissions.
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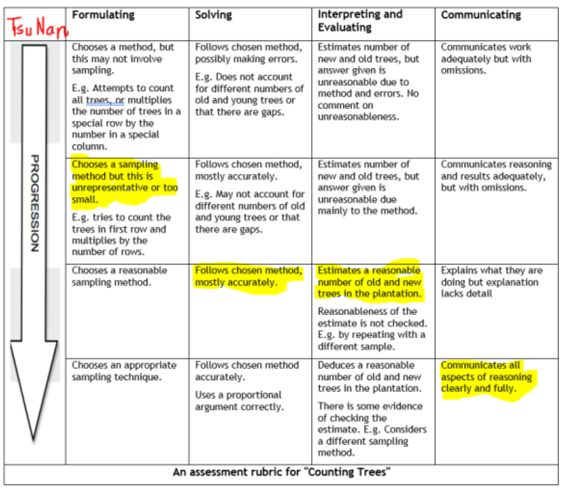
Chooses an appropriate sampling technique.	Follows chosen method accurately. Uses a proportional argument correctly.	Deduces a reasonable number of old and new trees in the plantation. There is some evidence of checking the estimate. E.g. Considers a different sampling method.	Communicates all aspects of reasoning clearly and fully.

An assessment rubric for "Counting Trees"

Completed rubric for Naima's work


TsuNan

Formulating	Solving	Interpreting and Evaluating	Communicating
Chooses a method, but this may not involve sampling. E.g. Attempts to count all trees, or multiplies the number of trees in a special row by the number in a special column.	Follows chosen method, possibly making errors. E.g. Does not account for different numbers of old and young trees or that there are gaps.	Estimates number of new and old trees, but answer given is unreasonable due to method and errors. No comment on unreasonableness.	Communicates work adequately but with omissions.
Chooses a sampling method but this is unrepresentative or too small. E.g. tries to count the trees in first row and multiplies by the number of rows.	Follows chosen method, mostly accurately. E.g. May not account for different numbers of old and young trees or that there are gaps.	Estimates number of new and old trees, but answer given is unreasonable due mainly to the method.	Communicates reasoning and results adequately, but with omissions.
Chooses a reasonable sampling method.	Follows chosen method, mostly accurately.	Estimates a reasonable number of old and new trees in the plantation. Reasonableness of the estimate is not checked. E.g. by repeating with a different sample.	Explains what they are doing but explanation lacks detail
Chooses an appropriate sampling technique.	Follows chosen method accurately. Uses a proportional argument correctly.	Deduces a reasonable number of old and new trees in the plantation. There is some evidence of checking the estimate. E.g. Considers a different sampling method.	Communicates all aspects of reasoning clearly and fully.

An assessment rubric for "Counting Trees"

Completed rubric for TsuNan's work

Involving students in self and peer assessment

Mathematical modelling, and the changed 'classroom contract' it requires, moves students into roles that the teacher ‘owns’ in traditional mathematics classrooms. Allowing students to become assessors of their and their peers’ work has several advantages including:

- an increased sense of agency, of ownership of and responsibility for the products of their reasoning,
- a liberation of the teacher from some of the challenges of detailed responsibility for the diversity of learning of the many students in their class.

This is especially the case for formative assessment, where the over-riding goal is helping students learn.

> “... self-assessment by students, far from being a luxury, is in fact an essential component of formative assessment. Where anyone is trying to learn, feedback about their efforts has three elements—the desired goal, the evidence about their present position, and some understanding of a way to close the gap between the two. All three must to a degree be understood by anyone before they can take action to improve their learning” (Black & Wiliam, 1998).

This is particularly true when the focus of the assessment is on processes like those in modelling. Many students do not understand the nature and importance of these processes in mathematics. If a student’s goal is only to get ‘the right answer’, then she will not attend to the deeper purposes of the lesson.

Here are some strategies for helping students to become more aware of modelling processes.

- *Use a poster or handout*
  Make a poster showing the generic list of modelling processes and display this on the classroom wall. Refer to this often, so that they become more aware that your goals for the lesson are for them to become more able to formulate, solve, interpret, evaluate and communicate.

- *Create task-specific hints*
  Before the lesson, prepare some task-specific hints that apply the generic processes to the particular problem in hand. When students are stuck, give them the appropriate hint either on paper or orally.

- *Ask students to assess provided samples of work against a rubric that highlights the processes*
  After students have worked on a task, present them with some (prepared), sample responses from other students. These solutions provide alternative strategies students may not have considered and may also contain errors. Ask students to rank order these solutions, along with their own response, giving explanations as to why they think one response is better than another. Several of these units use this strategy.

- *Ask students to assess each other’s work*
  After tackling a task in pairs, students exchange their work. Each pair of students is given the work of another pair. Students make suggestions for ways of improving each solution and stick these on the work using “sticky” notes. These comments are passed back to the originators, who must then produce a final, improved version based on the comments received. This is a more challenging strategy for the teacher than using prepared responses as the issues that arise will be less predictable.

- *Students interview each other about the processes they have used*
  When students have finished working on a task, ask them to get into pairs (not with members of their own group if it has been a group task). Each member of a pair interviews the other about their approach and the processes they have used while working on the task. The teacher may provide some pre-prepared questions to assist in this. After noting down the replies, students change roles. Suitable questions might be:
    - *What factors did you consider including in your model? Which did you finally include, and why?*
    - *How did you decide on your method?*
    - *How could this work be improved?*
    - *Is there still something you are confused by?*

Comments on the units

Each of the five units has a particular focus, each raising some specific pedagogical points. In this section we outline, for each unit:

- the modelling focus
- the core mathematical concepts
- specific aspects of the teaching.

The materials required are listed in the plan for each unit. In most units:

- students need copies of the student sheets, and a calculator. Several units recommend work with spreadsheets. Graph paper should be available on request.
- a data projector to use the provided slideshows is highly recommended. The slides are reproduced in the lesson plans at the point at which you should share them with your class.
- some internet videos are recommended to bring the real world situations to life.

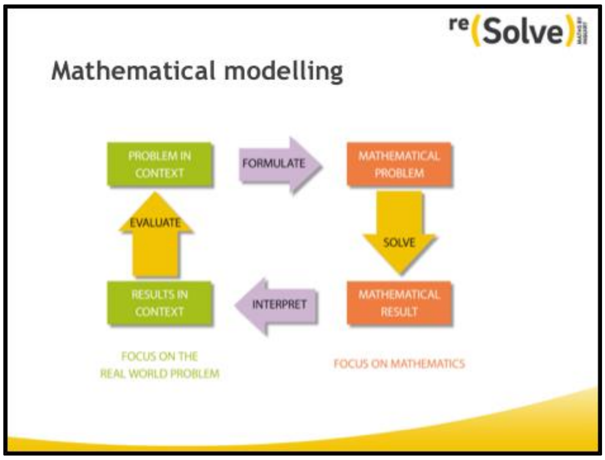
There is one overriding point to be made: the students will benefit from a good understanding of the mathematical concepts involved in a unit but, if you teach the topic shortly before the modelling unit, students will see it as an exercise – an *illustrative application* of the concept (see Figure 1) – not as a modelling problem. So, it is wise to **leave at least a few months between** teaching a topic and using a modelling task in which it plays the major role.

In some cases, students will formulate new mathematics themselves – tackling as a non-routine problem an example related to a topic that may be taught as a standard model in later years.

Unit 1 should be taught first. If this is not possible, use the Quick Start Lesson before any of the other units.

Unit 1: Introduction to mathematical modelling

It is ideal to start with this unit as it is designed to help students acquire a deeper understanding of mathematical modelling and the key processes involved, as summarised in the modelling diagram that is used throughout the modelling units.



Through engaging with the modelling of two familiar queuing situations – traffic jams and waiting in line at a theme park – students develop an overview of modelling at a meta-cognitive level. They:

- Take steps to understand the problem in context.
- Simplify the situation by making reasonable assumptions.
- Identify important variables and represent them and their relationships mathematically.
- Select appropriate mathematical methods to use.
- Explore the effects of systematically varying the constraints.
- Interpret and evaluate model predictions.
- Communicate their reasoning clearly.

The main pedagogical challenge here is to balance keeping students' attention on the modelling process while they are tackling substantial challenges in solving the actual problems *using mathematical concepts*.

Core mathematical concepts

- **Proportional reasoning** in space and time, with opportunities for translation into graphical and algebraic forms.
  This is a key area for modelling the real world. It is standard content but often only in the narrow form of short word problems in units on proportional reasoning – so the students already *know* that they will need to use ideas of proportional reasoning. Students need to be able to *recognise* proportionality in a practical situation, and to formulate the relationship mathematically.
- **Rates of flow**
  Rates of flow are familiar in practice but challenging mathematically with students often confusing relationships and quantities. This unit involves flow rates, such as cars per minute, or people per hour passing a point. It is often helpful to suggest to students that they work numerically, then express the

relationship in words, then develop this to use graphs and/or symbols. It is also helpful for students to consider the units for each of the quantities they deal with (e.g. cars per kilometre, km/h).

Unit 2: Pricing for Profit

This unit is about optimising within constraints, treated as a non-routine problem to solve. It is, of course, a standard topic that may be taught in later years. The unit presents a scenario of students making wooden animals and biscuits to sell for charity at a school fair. The goal is to work out what course of action will produce the greatest profit. In doing this, students:

- construct a very simple direct proportion model, assuming that profit depends only on the number of items sold, and the selling price,
- develop this model to consider the effect of increased sales price on number of sales, and hence on profit (a quadratic relationship),
- refine the model further, perhaps using a given spreadsheet, to enable a range of values for the parameters to be considered and to include other factors such as the cost of materials,
- consider processes for validation of the model, write a report on their work and critique other reports.

One lesson steps aside from the main work on the model development to consider the formulation stage in modelling in more detail. Here, a series of scenarios are presented that can be used to support students to identify variables, select and generate relationships between variables.

Through the lessons, students increase their understanding of the mathematical modelling process and link their work to the mathematical modelling diagram.

Core mathematical concepts

- Proportional reasoning in a money context – one that students normally find straightforward.
- Demand-Price relationship and concept of decreasing returns, which involves solving linear equations.
- Numerical and graphical methods and the recognition that they are essential for realistic modelling of this kind of situation.
- Creating and using formulas. This is mainly done using spreadsheets. Spreadsheets provide a powerful modelling tool in many situations. They provide a useful and natural means by which students can gain experience in creating and using formulas.

Images from Pricing for Profit Lesson 2

Unit 3: Packaging Designer

This unit gives students an experience of integrating issues associated with both design and mathematical modelling, and experience of working with the processes involved in each. Students think their way into the role of being a designer by considering the factors important in the design of packaging. They design a container to hold five cylindrical products (e.g. candles, bottles) with exact dimensions not known. They consider the priorities of the manufacturer (e.g. easily manufactured/engineered and transported) and the perspectives of customers (e.g. attractive package, pleasing shaped package etc.).

The lesson sequence is:

1. Understanding the context and developing initial plans and sketches.
2. Introducing design constraints by taking into account issues such as consideration of transport and manufacturing requirements (e.g. cutting from single sheet of card), and modifying plans.
3. Critiquing three reports on the packaging design.
4. Presenting students' own designs and reports, and thinking about the design and modelling cycles.

Mathematical approaches can range from direct drawing and measurement of diagrams, to use of trigonometry and algebra to express how the design can be adapted for any dimensions of the products.

One lesson steps aside to focus on the interpreting and evaluating stage of mathematical modelling. This lesson involves different content including reading graphs, proportional reasoning, and optionally trend lines.

Core mathematical concepts

- Connecting three-dimensional objects with their nets and other two-dimensional representations.
- Using appropriate units of measurement for area and volume and converting from one unit to another.
- Investigating the relationship between features of circles such as circumference, area, radius and diameter. Use formulas to solve problems involving circumference and area.
- Calculating the surface area and volume of cylinders and solve related problems.
- Investigating Pythagoras' Theorem and its application to solving simple problems involving right-angled triangles.
- Apply trigonometry to solve right-angled triangle problems.

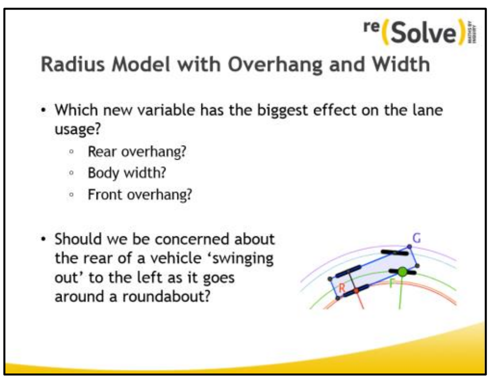
Unit 4: Cornering

This unit provides insight into the complex situation of long vehicles turning corners. The motion of turning vehicles is difficult to visualise and prone to misconceptions, and makes an excellent context for developing students' modelling skills. Students start from a very simple scale model and gradually build their understanding with help from an experiment and several pre-made dynamic geometry (Geogebra) models. This allows students to experience having to make sense of the mathematics of someone else: a skill that is very useful in the future in the workforce. Alternative models are provided, using different mathematical knowledge. With each model they find new questions to ask and consider how a different or improved model could provide the answers.

At the end of the unit, students give a presentation on road design with implications for road safety as well as for learning to drive. Communicating successfully to meet different audience's needs is a key learning goal.

The lesson sequence is:

1. Students discuss the meaning of the "Do not overtake turning vehicle" sign found on some trucks and buses. They explore with a paper scale model why a truck or bus needs a lot of space to turn.
2. Students walk a bicycle/scooter/toy in an arc, discovering that the rear wheel travels along an arc of smaller radius than the front wheel and explore the associated geometry.
3. Using software models (supplied), students examine how much the rear wheels cut in for a variety of vehicles, considering vehicle length, width and finally front and back overhangs.
4. Students use their best models to make recommendations related to roundabout or car park design.



Core mathematical concepts

- Features of circles such as circumference, radius, diameter, tangents.
- Pythagoras' Theorem and its application to solving simple problems involving right angled triangle.
- Apply angle and chord properties of circles.

Unit 5: How risky is life?

This unit introduces 'descriptive modelling' – the modelling of situations where data plays the central role. Data is sourced from Australian Bureau of Statistics official reports. The unit involves data analysis and statistical modelling, with a strong focus on the interpretation of data and evaluation of the inferences drawn from it. Through modelling the risk of dying in the coming year and the relative probability of various causes of death - unnatural and natural – students learn about the limits and dangers of anecdotes and impressions, the attributes of reliable data, and expected degrees of random variation, and the need to interpret data very carefully by considering multiple explanations of apparent trends. Throughout the goal is to develop students' ability to work and think independently – and to learn that:

- Life is risky but, in our society, not very.
- Mathematical thinking is essential for getting risks in perspective – and that it is orders of magnitude, rather than precise numbers, that are significant.

The unit starts with students bringing out the mismatch between real and perceived risk. They explore the risks of dying unexpectedly by various causes. They start from the fears they know and, by comparing them with real-life data, they recognize that these are often unfounded and driven by the media. Students learn how to calculate the risks involved for various activities and how these are related to the base risk of death for typical people of different ages and genders. The emphasis is on order-of-magnitude comparisons, reflecting the various kinds of variation in risk level between individuals and over time.

In the unit students will:

- Define a problem situation of interest (risk).
- Focus down to some specific questions (initially, risks of death from unnatural causes).
- Formulate the questions in mathematical terms – identifying possible causes.
- Solve by collecting appropriate data, then represent it in ways that help reveal its meaning.
- Interpret the data, considering the influence of hidden factors, and drawing inferences about the context.
- Evaluate and critique both the data and the inferences.
- Communicate their reasoning clearly.

These steps are those of the modelling cycle in a data-driven context like this. The modelling cycle is repeated as the model is improved to include all causes of death and how they depend on age and gender.

Core mathematical concepts

- Data location and collection, organization and presentation in various forms
- Order of magnitude reasoning, with very large and very small numbers
- Drawing inferences from data carefully
- Statistical, and logical limits, on such inferences
- Random variation and how large it might be.

Optional 'quick start' lesson

This single lesson prompts students to consider carefully the process of mathematical modelling. It emphasises the different aspects of formulating, interpreting and evaluating a model in a situation that will hopefully prove not too demanding in terms of the mathematics. The lesson involves using ideas of distance, speed and time in the context of bushwalking and growth in the context of a family having children.

In the context of bushwalking students are encouraged to work using whatever is comfortable, expressing rules in words or symbols. They consider how they would estimate how long a bushwalk would take and understand this as a problem of modelling a real world situation. They consider how they might improve their model by including some factors that were ignored at first.

Discussion highlights the modelling processes involved and these are then re-experienced in a new context in which students explore how the number of descendants of someone might grow over several generations. Again, a very simple model is advised to start with and if time permits this might be developed further.

Core mathematical concepts

- Speed, distance, time considered in a simple context with relationships expressed in words or symbols.
- Opportunities to extend to graphical representations.
- Simple ideas of exponential growth and associated numerical calculations.

Acknowledgements and Further information

Acknowledgements

These materials were designed for *reSolve* by a Shell Centre team consisting of Professor Hugh Burkhardt, Rita Crust, Daniel Pead and Professor Geoff Wake from the University of Nottingham, U.K. It draws on their extensive experience in creating student materials to foster mathematical modelling, and in designing resources to help teachers prepare for teaching mathematical modelling. Some parts of this guide are based on earlier Shell Centre materials.

The work was commissioned and edited by Kaye Stacey and Lucy Bates of the reSolve: Maths by Inquiry team.

References and Further Reading

Alexander, R.J. (2017) *Towards Dialogic Teaching: rethinking classroom talk* (5th edition), Dialogos.

Black, P., Harrison, C., Lee, C., Marshall, B., & Wiliam, D. (2003). Assessment for learning: Putting it into practice. Buckingham: Open University Press.

Bowland Charitable Trust (2007). *Bowland Mathematics* http://www.bowlandmaths.org.uk/

Brousseau, G. (1990). Le contrat didactique et le concept de milieu: Dévolution. *Recherches en didactique des Mathématiques* 9 (3), 309-336

Burkhardt, H. (2017). Ways to teach modelling – a 50 year study, ZDM DOI 10.1007/s11858-017-0899-8 Available at: https://link.springer.com/content/pdf/10.1007%2Fs11858-017-0899-8.pdf

Hirsch, C. R., Ed.; McDuffie, A. R., (Eds.). Annual Perspectives in Mathematics Education 2016: Mathematical Modelling and Modeling Mathematics. National Council of Teachers of Mathematics.

Hodgen, J., & Wiliam, D. (2006). Mathematics inside the black box. (Part of 'Black Box' assessment series). GL Education Group. https://gl-education.com/ and https://gl-education.com/products/black-box-series/.

Lowe, J., Carter, M., & Cooper, T. (2018). Mathematical modelling in the junior secondary years: An approach incorporating mathematical technology. *Australian Mathematics Teacher* 74(1), 4 – 12.

Mathworks Math Modeling Challenge (M3C). Math Modeling and Teaching Math Modeling Video Series. https://m3challenge.siam.org/resources/teaching-modeling-videos

Mercer, N. (2012). What has the study of classroom talk told us that can improve the quality of education. https://www.slideshare.net/margarubiosoto/exploratory-talk-in-professor-neil-mercer

Pólya, George *(1957). How to Solve It. Garden City, NY: Doubleday.*

Steen, L. A. (ed.) (2002). *Mathematics and Democracy: the case for quantitative literacy,* Washington, DC: National Council on Education and the Disciplines, http://www.maa.org/ql/mathanddemocracy.html

Stillman. G., Galbraith, P., Brown, J., & Edwards, I. (2007). A Framework for Success in Implementing Mathematical Modelling in the Secondary Classroom. In J. Watson and K. Beswick (Eds) *Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia*. MERGA. https://www.aamt.edu.au/Media/Connect-with-Maths-in-Action/Professional-Reading/Framework-for-Modelling-Secondary/(language)/eng-AU

Usiskin, Z. (2011). Mathematical Modeling in the School Curriculum. http://ucsmp.uchicago.edu/resources/conferences/2011-09-11/

Further resources for teaching

Angelini, M., Coupland, M., & Prescott, A. et al. (2016). *Maths Inside: Highlighting the Role of Mathematics in Society as Motivation to Engage More in Mathematical Activities.* https://www.aamt.edu.au/Better-teaching/Classroom-resources/Maths-Inside (A set of 8 Australian resources for Years 7 – 10 approximately. Topics include "Big data, better hospitals", and "Prawns for Profit". Classroom resources including videos.

Consortium for Mathematics and Its Applications. http://www.comap.com/
Two modelling competitions and a huge number of free and commercial classroom resources.

International Mathematical Modeling Challenge, IM^{2}C. **http://immchallenge.org/Index.html**
A student competition, with many resources for support.

Mathworks Math Modeling Challenge (M3C) https://m3challenge.siam.org/resources
A student competition, and many resources for support, including a good set of videos on modelling.