Quick Guide

This is a shortened version of the Teachers Guide, especially for participants in professional learning sessions. Please consult *ST5_Reasoning_TeachersGuide.docx* to get a full picture of these resources.

What is Reasoning?

Reasoning is one of the proficiencies in the Australian Curriculum: Mathematics. "It plays a critical role in developing students' understanding and promoting creative thinking in mathematics." (Vale, et al., 2017, p. 3). Students discover and make sense of mathematical ideas and concepts when engaged in inquiry-based tasks that require them to reason.

"Reasoning is the glue that holds everything together, the lodestar that guides learning" (Kilpatrick et al., 2001, p. 129).

Reasoning in the Australian Curriculum: Mathematics is defined as:

> *Students develop an increasingly sophisticated capacity for logical thought and actions, such as **analysing**, **proving**, **evaluating**, **explaining**, **inferring**, **justifying** and **generalising**. Students are reasoning mathematically when they **explain** their thinking, when they **deduce** and **justify** strategies used and conclusions reached, when they **adapt** the known to the unknown, when they **transfer** learning from one context to another, when they **prove** that something is true or false and when they **compare** and **contrast** related ideas and **explain** their choices.*

Three Key Reasoning Actions

These resources divide mathematical reasoning into three main reasoning actions: *Analysing*, *Generalising*, and *Justifying*. Together, these three actions cover all the components of reasoning described in the Australian Curriculum. The learning trajectories show how these three reasoning actions develop.

Analysing

Analysing involves ***exploring*** the problem using examples provided or generating examples to form or test a conjecture about a common property, pattern or relationship.

Analysing occurs by ***comparing and contrasting*** cases to notice:

- what is same and what is different, and to sort and classify the cases.
- what stays the same and what changes and to recall, repeat or extend the pattern.

Analysing involves using ***numerical or spatial structure***, ***known facts or properties*** when sorting cases or repeating and extending pattern.

Categories of cases and patterns are identified by *labelling* using ***terms, diagrams or symbols***

We value your feedback after these tasks via https://www.surveymonkey.com/r/RJC6FPC

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Generalising

Generalising involves forming conjectures that is, ***developing statements*** that are thought to be true but not yet known or shown to be true.

Generalising involves ***identifying common*** *properties* or patterns across more than one case and ***communicating a rule (conjecture)*** to describe the common property, pattern or relationship.

The ***statement or rule is communicated*** orally or written using words, diagrams or symbols.

The ***meaning of the statement or rule is communicated*** using ***particular examples*** to explain the property or pattern. Further examples are used to explain how the rule applies to other cases, that is to show that how it is a generalisation.

Justifying

Justifying involves ***checking the truth of conjectures and generalisations*** to demonstrate or refute the truth of a claim.

Justifying uses ***logical argument*** to convince others of the truth of the claim or to refute the claim.

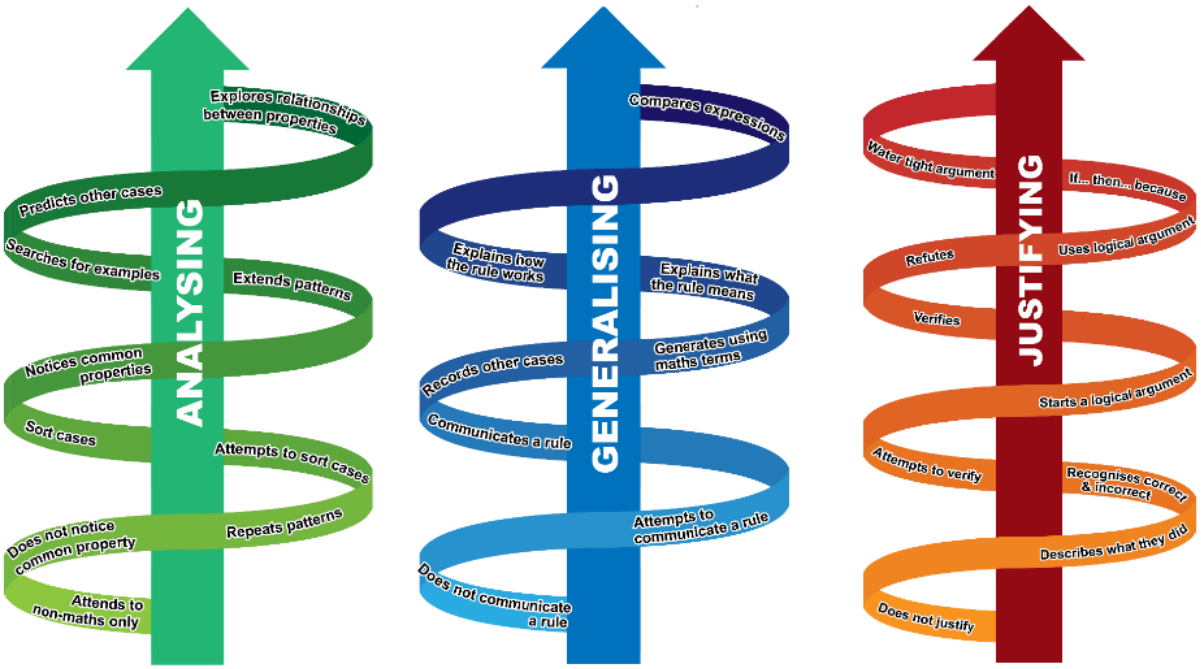
A ***logical argument*** is made by:

- using ideas that are already understood;
- following agreed processes or steps for making arguments; and
- using terms, diagrams and symbols that are known and understood.

A ***mathematical refutation*** involves demonstrating that a particular statement is false.

Tasks are usually approached by first analysing, then generalising, then justifying. Some tasks provide a conjecture to prove or disprove, and so the generalising is already done.

There are three learning trajectories that show how students typically progress in each key reasoning action.



Mathematical Reasoning Rubric

When conducting formative assessment of reasoning, teachers are aiming to find out how students analyse, generalise and justify and how they communicate their findings and argument.

The Mathematical Reasoning Rubric is a ready-to-use formative assessment tool designed to identify both the type of reasoning action that the student is using and the level of proficiency that the student is demonstrating irrespective of the mathematical content of the task.

There are five levels of proficiency for each key reasoning action: not evident, beginning, developing, consolidating and extending. These steps do not align with a year level. Student may not be at the same level for each reasoning action. Using enabling and extending prompts will encourage students to develop proficiency with analysing and demonstrate a higher level of analysing.

The rubric can be used:

- To assess individual students, or groups of students, or to plan learning for a whole class.
- With observations during a lesson, or by analysing written or recorded student work after a lesson.
- Alone, or collaboratively with your colleagues.
- With the exemplar tasks, or with your own tasks.

Observation of student's reasoning:

	Analysing	Generalising	Justifying
NOT EVIDENT	• Does not notice common property or pattern.	• Does not communicate a common property or rule (conjecture) for a pattern.	• Does not justify.
BEGINNING	• Recalls random known facts or attempts to sort examples or repeats patterns.	• Attempts to communicate a common property or rule (conjecture) for a pattern.	• Describes what they did and recognises what is correct or incorrect. • Argument is not coherent or does not include all steps.
DEVELOPING	• Notices a common property, or sorts and orders cases, or repeats and extends patterns. • Describes the property or pattern.	• Generalises: communicates a rule (conjecture) using mathematical terms, and records other cases or examples.	• Attempts to verify by testing cases, and detects and corrects errors or inconsistencies. • Starting statements in a logical argument are correct.
CONSOLIDATING	• Systematically searches for examples, extends patterns, or analyses structures, to form a conjecture. • Makes predictions about other cases.	• Generalises: communicates a rule (conjecture) using mathematical symbols and explains what the rule means or explains how the rule works using examples.	• Verifies truth of statements by confirming all cases or refutes a claim by using a counter example. • Uses a correct logical argument.
EXTENDING	• Notices and explores relationships between properties.	• Generalises cases, patterns or properties using mathematical symbols and applies the rule. • Compares different expressions for the same pattern or property to show equivalence.	• Uses a watertight logical argument. • Verifies that the generalisation holds for *all* cases using logical argument.

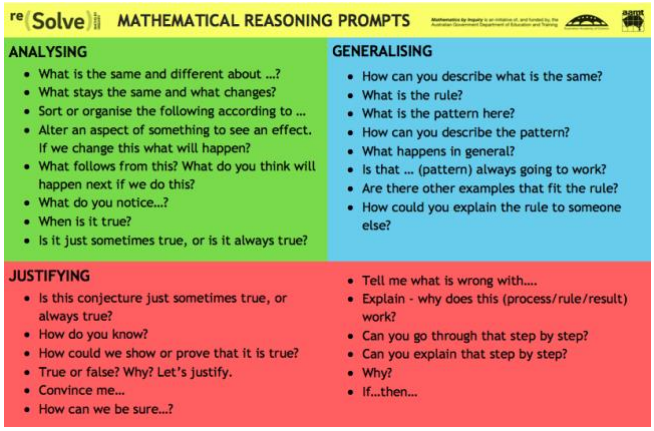
Comments (feedback, reasoning prompts for further development):

Prompts

Teachers may need to use prompts to elicit students' reasoning. Examples are included in the chart below. A poster for classroom use is included in this Quick Guide and a colour coded set of reasoning prompt cards which can be laminated and then used across may lessons is provided in the full Teacher' Guide.

Three main types of prompts are suggested in the lessons:

- Reasoning prompts - to highlight the general questions about reasoning that students can consider across many problems;
- Enabling Prompts – to help students get started on a problem;
- Extending Prompts – to suggest how students who find the task easy can get more out of it.



Sample assessment

A Magic V is a number puzzle. The aim is to arrange five consecutive numbers so that the sum of each 'arm' of the V is the same. Below are two Vs. The left V is a Magic V (4+2+3=9 and 5+1+3=9); the V on the right is not. Each number can be used only once.

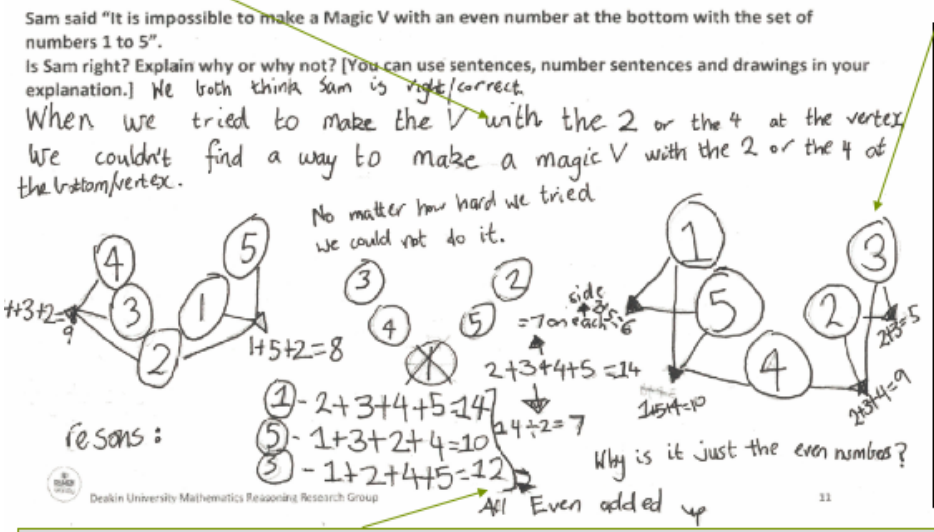
Sam said, "It is impossible to make a Magic V with an even number at the bottom with the set of numbers 1 to 5."

Is Sam right? Explain why or why not.

JUSTIFYING: Starting statements in a **logical argument** are correct and accepted by the class.

ANALYSING: Sorts and classifies according to a common property.

The student is analysing the total of the arms and sorting according to even and odd numbers at the vertex.



ANALYSING: Developing

GENERALISING: Beginning

JUSTIFYING: Developing

Teacher Prompt:

You have made a very interesting remark that if we have an even number at the bottom then the total of the two arms are not equal.

Is it true for all different Magic V with 2 and 4 at the bottom? Why?

What do we need to have at the bottom to make the two arms equal?

JUSTIFYING: Attempts to verify by testing cases or explaining the meaning of a conjecture using one example.

GENERALISING: Detecting and correcting errors and inconsistencies using materials, diagrams and informal written methods.

The next step is for the student to think about the properties of odd and even numbers.

Student Name: Work Sample 2 Reasoning Task: MAGIC V Date:

Observation of student's reasoning:

* Adding total of arms & comparing 'Vs' with an odd/even number at vertex.
* developing a logical argument but no 'why'

	Analysing	Generalising	Justifying
Not Evident	• **Does not notice** common property or pattern.	• **Does not communicate** a common property or rule (conjecture).	• **Does not justify**.
Beginning	• **Recalls** random known facts or **attempts to sort** examples or **repeats** patterns.	• **Attempts to communicate** a common property or rule for the pattern.	• **Describes** what they did and ✓ **recognises** what is correct or incorrect. • **Argument** is not coherent or does not include all steps.
Developing	• **Notices** a common property, or **sorts and orders** cases, or **repeats and extends** patterns. • **Describes** the property or pattern.	• **Generalises: communicates** a rule (conjecture) using mathematical terms and records other cases or examples.	• **Attempts to verify** by testing cases and **detects and corrects** errors or inconsistencies. • ✓ Starting statements in a **logical argument** are correct.
Consolidating	• **Systematically searches** for examples, extends pattern or analyses structure to form a conjecture. • Makes **predictions** about other cases. * 'what will..'	• **Generalises: communicates** a rule using mathematical symbols and **explains what** the rule means or **explains how** the rule works using examples.	• **Verifies** truth of statements by confirming all cases or **refutes** a claim by using a counter example. * • Uses a correct **logical argument.**
Extending	• **Notices and explores** relationships between properties.	• **Generalises** cases, patterns or properties using mathematical symbols (including algebraic symbols) and **applies** the rule. • **Compares** different expressions for the same pattern or property to show equivalence.	• Uses a **watertight logical argument**. • **Verifies** that the generalisation holds for *all* cases using logical argument.

Comments (feedback, reasoning prompts for further development):

* Look closer at properties of odd/even numbers
* Develop 'justification' → look at reasoning prompts.

Draft Levels of Reasoning & Rubric, Mathematical Reasoning Research Group, Deakin University & ReSolve ST5, AAMT & Academy of Science, December, 2018

Summary of Exemplars

The Exemplars in this resource show how teachers have used the Rubric for formative assessment of students' reasoning. The annotated work samples included in each Exemplar show the evidence that teachers used.

Most tasks can be easily adapted for students across Years 3 to 6 and sometimes lower. The year levels specified indicate the year level of the students who provided the sample work.

Exemplar: Is it True? (Year 3)

This task asks students to decide whether an addition is correct or not and to explain their reasons. The content focus is on place value and grouping, re-grouping and renaming for multi-digit addition. The reasoning foci of this task are explaining and justifying statements; exploring and noticing relationships (analysing); and forming conjectures and generalising. The task is easy to adapt to other content and year levels.

Exemplar: Number Towers (Year 3)

Number Towers gives students an opportunity to develop and test conjectures and form generalisations by reasoning mathematically about numerical structures with addition. The task promotes careful analysis of a mathematical structure. Students need to experiment systematically, keep track of results, and choose cases carefully to test the rule. The task can be adapted for older students by using fractions or decimals.

Exemplar: The Magic V (Year 4)

The Magic V task affords an opportunity to explain the reasons why a conjecture is true. Students begin by creating Magic Vs by trial and error, but come to see constant features. By manipulating numbers, analysing totals and recording their ideas, students will come to see why the properties of odd and even numbers are relevant. The main purpose is to develop students' capacity to analyse situations, to find reasons and develop logical arguments.

Exemplar: Matchsticks (Year 4)

Students explore making rows of squares from matchsticks. The patterns can be described pictorially, numerically and symbolically. Students learn to work systematically and keep a record of results that assist them to develop and test conjectures. As the students describe and explain patterns, they will begin to move from additive to multiplicative reasoning. By changing from rows of squares, the task is easy to adapt to other year levels.

Exemplar: Shapeshifter (Year 4)

Shapeshifter is based on a picture story book. Students find a rule for using a straight line to dissect a polygon to make a polygon with one more side. The content foci are naming shapes and their components. The reasoning foci are especially Analysing and Justifying – seeing and explaining when and why cutting off a corner increases the number of sides.

Exemplar: What Else Belongs? (Year 5)

In this task, students find common properties of three given numbers. They notice and describe properties of number; such as size, order, composition, place value, multiples, factors, even or odd. The students justify why some numbers have similar or dissimilar properties. There is a strong emphasis on Analysing – seeing what is the same and what is different. The task can very readily be adapted to other content and year levels.

Exemplar: Area and Perimeter (Year 6)

In this task, students respond to a conjecture related to a common misconception: that a rectangle with a larger perimeter will always have a larger area. Students will learn that it is sufficient to offer one counter example to refute a conjecture or general statement that makes a claim about all cases.

Exemplar: Painted Cube (Year 6)

The Painted Cube task is rich and complex, providing students with opportunities to explore a variety of patterns that can be described spatially, numerically and symbolically. There are good opportunities for using visualisation. Students learn to work systematically by keeping a clear record of results which will encourage them to develop and test conjectures and to ask themselves questions about further cases.

Student Name: **Reasoning Task:** **Date:**

Observation of student's reasoning:

	ANALYSING	GENERALISING	JUSTIFYING
NOT EVIDENT	• **Does not notice** common property or pattern.	• **Does not communicate** a common property or rule (conjecture) for a pattern.	• **Does not justify**.
BEGINNING	• **Recalls** random known facts or **attempts to sort** examples or **repeats** patterns.	• **Attempts to communicate** a common property or rule (conjecture) for a pattern.	• **Describes** what they did and **recognises** what is correct or incorrect. • A**rgument** is not coherent or does not include all steps.
DEVELOPING	• **Notices** a common property, or **sorts and orders** cases, or **repeats and extends** patterns. • **Describes** the property or pattern.	• **Generalises: communicates** a rule (conjecture) using mathematical terms, and records other cases or examples.	• **Attempts to verify** by testing cases, and **detects and corrects** errors or inconsistencies. • Starting statements in a **logical argument** are correct.
CONSOLIDATING	• **Systematically searches** for examples, extends patterns, or analyses structures, to form a conjecture. • Makes **predictions** about other cases.	• **Generalises: communicates** a rule (conjecture) using mathematical symbols and **explains what** the rule means or **explains how** the rule works using examples.	• **Verifies** truth of statements by confirming all cases or **refutes** a claim by using a counter example. • Uses a correct **logical argument.**
EXTENDING	• **Notices and explores** relationships between properties.	• **Generalises** cases, patterns or properties using mathematical symbols and **applies** the rule. • **Compares** different expressions for the same pattern or property to show equivalence.	• Uses a **watertight logical argument**. • **Verifies** that the generalisation holds for *all* cases using logical argument.

Comments (feedback, reasoning prompts for further development):

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Assessing Mathematical Reasoning: Rubric

This table elaborates the different levels of proficiency for each key reasoning action. It provides more detail than the classroom assessment sheet, and should be used as a guide when looking at student work.

The levels reflect the learning trajectories. There are several descriptors for each level, for each key reasoning action, but not all descriptors need to be present for a student to be at that level. One reason is that different tasks bring out different reasoning actions. For example: *What Else Belongs?* requires verification and logical argument whereas *Area and Perimeter* requires refutation using a counter example (although there are opportunities for other reasoning in the follow up tasks). The exemplars provide examples of students' reasoning matched to descriptors in this rubric.

	ANALYSING	GENERALISING	JUSTIFYING
NOT EVIDENT	• **Does not notice** numerical or spatial structure of examples or cases. • Attends to **non-mathematical** aspects of the examples or cases.	• **Does not communicate** a common property or rule for a pattern.	• **Does not justify**. • **Appeals** to teacher or others.
BEGINNING	• **Notices** similarities across examples • **Recalls** random known facts related to the examples. • Recalls and **repeats patterns** displayed visually or through use of materials. • Attempts to **sort** cases based on a common property.	• **Draws attention to** or **attempts to communicate** a common property or repeated components of a pattern using: ○ body language (gesture), ○ drawing, ○ concrete materials ○ counting or • oral language (metaphors).	• **Describes** what they did and why it may or may not be correct. • **Recognises** what is correct or incorrect using materials, objects, or words. • Makes **judgements** based on simple criteria such as known facts. • The **argument may not be coherent** or include all steps in the reasoning process.
DEVELOPING	• **Notices** a common numerical or spatial property. • **Recalls** and **repeats** patterns using numerical structure or spatial structure. • **Sorts and classifies** cases according to a common property. • **Orders** cases to show what is the same or stays the same and what is different or changes. • **Describes** the case or pattern by labelling the category or sequence.	• **Communicates** a rule (conjecture) about a: ○ *property* using words, diagrams or number sentences. ○ *pattern* using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition. • **Records** other cases that fits the rule (conjecture) or extends the pattern using the rule.	• **Attempts to verify** by testing cases or explaining the meaning of a conjecture using one example. • **Detecting and correcting errors** and inconsistencies using materials, diagrams and informal written methods. • Starting statements in a **logical argument** are correct and accepted by the classroom.

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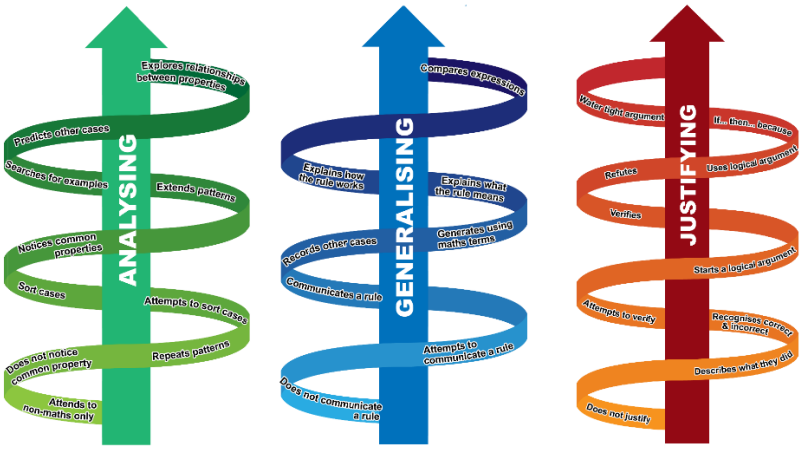
CONSOLIDATING	• **Notices more than one** common property by systematically generating further cases and/or listing and considering a range of known facts or properties.
	• **Repeats and extends patterns** using both the numerical and spatial structure.
	• **Searches for and produces examples**:
	  ○ using tools, technology and modelling
	• Makes **predictions** about other cases:
	  ○ with the same property
	  ○ included in the pattern
		• **Generalises: communicates a rule (conjecture)** using mathematical terms, symbols or diagrams (e.g. a number sentence or labelled geometric diagram)
		• **Explains what** the rule (conjecture) means using one example.
		• **Extends** the pattern using an example to **explain how** the rule works.
			• **Verifies** truth of statements by using a common property, rule or known facts that confirms each case. May use materials and informal methods.
			• **Refutes** a claim by using a counter example.
			• Uses a correct **logical argument** that has a complete chain of reasoning and uses words such as 'because', 'if…then…', 'therefore', 'and so', 'that leads to'
			• **Extends the generalisation** using logical argument.

EXTENDING	• **Notices and explores** relationships between:
	  ○ common properties
	  ○ numerical structures of patterns.
		• **Generalises: communicates** the rule (conjecture) using mathematical symbols.
		• **Applies** the rule to find further examples or cases.
		• **Generalises** properties by forming a statement about the relationship between common properties.
		• **Compares** different symbolic expressions used to define the same pattern to show equivalence.
			• Uses a **watertight logical argument** that is mathematically sound and leaves nothing unexplained.
			• **Verifies** that the statement is true or the generalisation holds for *all* cases using logical argument.



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MATHEMATICAL REASONING PROMPTS

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ANALYSING

- What is the same and different about …?
- What stays the same and what changes?
- Sort or organise the following according to …
- Alter an aspect of something to see an effect. If we change this what will happen?
- What follows from this? What do you think will happen next if we do this?
- What do you notice…?
- When is it true?
- Is it just sometimes true, or is it always true?

GENERALISING

- How can you describe what is the same?
- What is the rule?
- What is the pattern here?
- How can you describe the pattern?
- What happens in general?
- Is that … (pattern) always going to work?
- Are there other examples that fit the rule?
- How could you explain the rule to someone else?

JUSTIFYING

- Is this conjecture just sometimes true, or always true?
- How do you know?
- How could we show or prove that it is true?
- True or false? Why? Let's justify.
- Convince me…
- How can we be sure…?
- Tell me what is wrong with….
- Explain - why does this (process/rule/result) work?
- Can you go through that step by step?
- Can you explain that step by step?
- Why?
- If…then…