ASSESSING MATHEMATICAL REASONING

Teachers' Guide

Contents

Students "actively engaged in making and testing generalisations can enjoy the joy of discovering a pattern and sharing it" (Stacey, 2012, p. 16-17).

"If students are consistently expected to explore, question, conjecture and justify their ideas, they learn that mathematics should make sense rather than believing that mathematics is a set of arbitrary rules and formulas." (Reys, et al., 2012, p. 97)

"(They're) connecting their understanding and their knowledge … they're bringing in all of their knowledge and they're putting it all together and figuring bigger things out through reasoning" (Year 3/4 Primary school teacher).

This Teachers' Guide introduces you to the reSolve Assessing Mathematical Reasoning resources. The resources assist primary teachers to encourage mathematical reasoning in their students and to conduct formative assessment of this proficiency, in line with the Australian Curriculum: Mathematics.

The resources include:

- A rubric based on three different reasoning actions that shows how students progress through different levels of proficiency. This is summarised in an assessment sheet for teachers to use in the classroom.
- Eight exemplar reasoning tasks ready for classroom use, with student solutions and sample teacher assessments.
- Reasoning prompt cards and poster that can be used to encourage reasoning in any mathematics lesson.
- Professional learning materials, for individuals or groups including a slide show (with optional audio).

After reading this guide we hope you will have a good idea of the different types of reasoning that are part of mathematics, how they can be developed in your classroom, and how you can use these resources for professional learning.

We value your feedback via https://www.surveymonkey.com/r/RJC6FPC

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

What is Reasoning?

Reasoning is one of the proficiencies in the Australian Curriculum: Mathematics. “It plays a critical role in developing students’ understanding and promoting creative thinking in mathematics.” (Vale, et al., 2017, p. 3). Students discover and make sense of mathematical ideas and concepts when engaged in inquiry-based tasks that require them to reason.

“Reasoning is the glue that holds everything together, the lodestar that guides learning” (Kilpatrick et al., 2001, p. 129).

Reasoning in the Australian Curriculum: Mathematics is defined as:

> *Students develop an increasingly sophisticated capacity for logical thought and actions, such as **analysing**, **proving**, **evaluating**, **explaining**, **inferring**, **justifying** and **generalising**. Students are reasoning mathematically when they **explain** their thinking, when they **deduce** and **justify** strategies used and conclusions reached, when they **adapt** the known to the unknown, when they **transfer** learning from one context to another, when they **prove** that something is true or false and when they **compare** and **contrast** related ideas and **explain** their choices.*

For a more detailed look at mathematical reasoning, see More on Reasoning in this document.

Three Key Reasoning Actions

These resources divide mathematical reasoning into three main reasoning actions: *Analysing*, *Generalising*, and *Justifying*. Together, these three actions cover all the components of reasoning described in the Australian Curriculum.

Analysing

Analysing includes:

- Exploring the problem and connecting with known facts and properties;
- Comparing and contrasting cases;
- Sorting and classifying cases.

Generalising

Generalising includes:

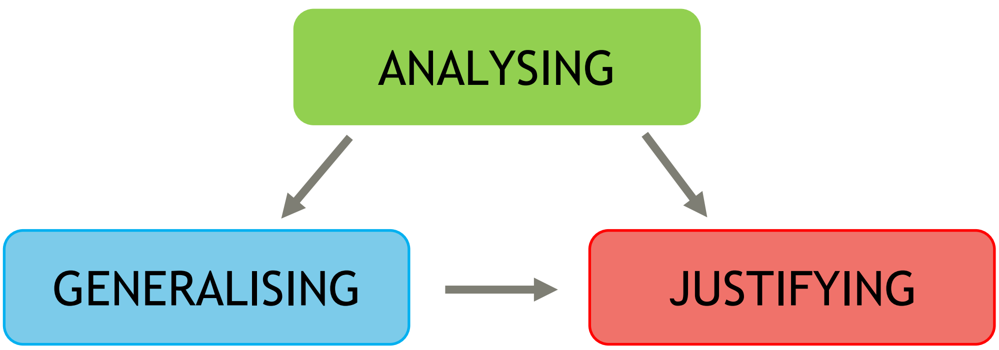
- Identifying common properties or patterns across cases;
- Forming conjectures, i.e. statements that are thought to be true but not yet known to be true;
- Communicating conjectures clearly.

Justifying

Justifying includes:

- Checking the truth of conjectures;
- Using logical argument to convince others;
- Refuting a claim.

Tasks are usually approached by first analysing, then generalising, then justifying. Some tasks provide a conjecture to prove or disprove, and so the generalising is already done.



Connecting the Key Reasoning Actions

Assessing Reasoning

Through detailed study of student work, the Mathematical Reasoning Research Group of Deakin University has developed a rubric that shows how each reasoning action develops. The rubric identifies five levels of increasing proficiency for each key reasoning action: not evident, beginning, developing, consolidating and extending. The descriptions of the levels are written to cover typical progression in the primary years. Note, however, that a level does not align with a school year, and a student may not be at the same level for all reasoning actions.

The rubric is presented in two ways. There is a full version (2 pages), and a summarised assessment sheet for class use (also shown below). Visual summaries of the learning trajectories are on the next pages: Analysing, Generalising and Justifying. A sample assessment follows.

The rubric can be used:

- To assess individual students, or groups of students, or to plan learning for a whole class.
- With observations during a lesson, or by analysing written or recorded student work after a lesson.
- Alone, or collaboratively with your colleagues.
- With the exemplar tasks, or with your own tasks.

Observation of student's reasoning:

	Analysing	Generalising	Justifying
NOT EVIDENT	• **Does not notice** common property or pattern.	• **Does not communicate** a common property or rule (conjecture) for a pattern.	• **Does not justify**.
BEGINNING	• **Recalls** random known facts or **attempts to sort** examples or **repeats** patterns.	• **Attempts to communicate** a common property or rule (conjecture) for a pattern.	• **Describes** what they did and **recognises** what is correct or incorrect. • **Argument** is not coherent or does not include all steps.
DEVELOPING	• **Notices** a common property, or **sorts and orders** cases, or **repeats and extends** patterns. • **Describes** the property or pattern.	• **Generalises: communicates** a rule (conjecture) using mathematical terms, and records other cases or examples.	• **Attempts to verify** by testing cases, and **detects and corrects** errors or inconsistencies. • Starting statements in a **logical argument** are correct.
CONSOLIDATING	• **Systematically searches** for examples, extends patterns, or analyses structures, to form a conjecture. • Makes **predictions** about other cases.	• **Generalises: communicates** a rule (conjecture) using mathematical symbols and **explains what** the rule means or **explains how** the rule works using examples.	• **Verifies** truth of statements by confirming all cases or **refutes** a claim by using a counter example. • Uses a correct **logical argument.**
EXTENDING	• **Notices and explores** relationships between properties.	• **Generalises** cases, patterns or properties using mathematical symbols and **applies** the rule. • **Compares** different expressions for the same pattern or property to show equivalence.	• Uses a **watertight logical argument**. • **Verifies** that the generalisation holds for *all* cases using logical argument.

Comments (feedback, reasoning prompts for further development):

Analysing

Analysing involves ***exploring*** the problem using examples provided or generating examples to form or test a conjecture about a common property, pattern or relationship.

Analysing occurs by ***comparing and contrasting*** cases to notice:

- what is the same and what is different, and to sort and classify the cases.
- what stays the same and what changes and to recall, repeat or extend the pattern.

Analysing involves using ***numerical or spatial structure***, ***known facts or properties*** when sorting cases or repeating and extending patterns.

Categories of cases and patterns are identified by ***labelling*** with ***words or symbols***, or showing with ***diagrams***.

As students develop their capacity to analyse they become more systematic and more often notice things that matter. They draw on a broader range of their prior knowledge to generate examples to search for a common property, pattern or relationship.

The learning trajectory for analysing (shown opposite) represents the different levels of proficiency a student might display. Analysing that is higher up the trajectory will have explored more ways in which cases can be compared and contrasted, and looked for deeper common properties and connections with known facts or properties.

It is important to provide opportunities for students to focus on analysing. Using reasoning prompts like those below will encourage students at any year level to develop their analysing proficiency.

What is the same and different about …?

What stays the same and what changes?

Sort or organise the following according to …

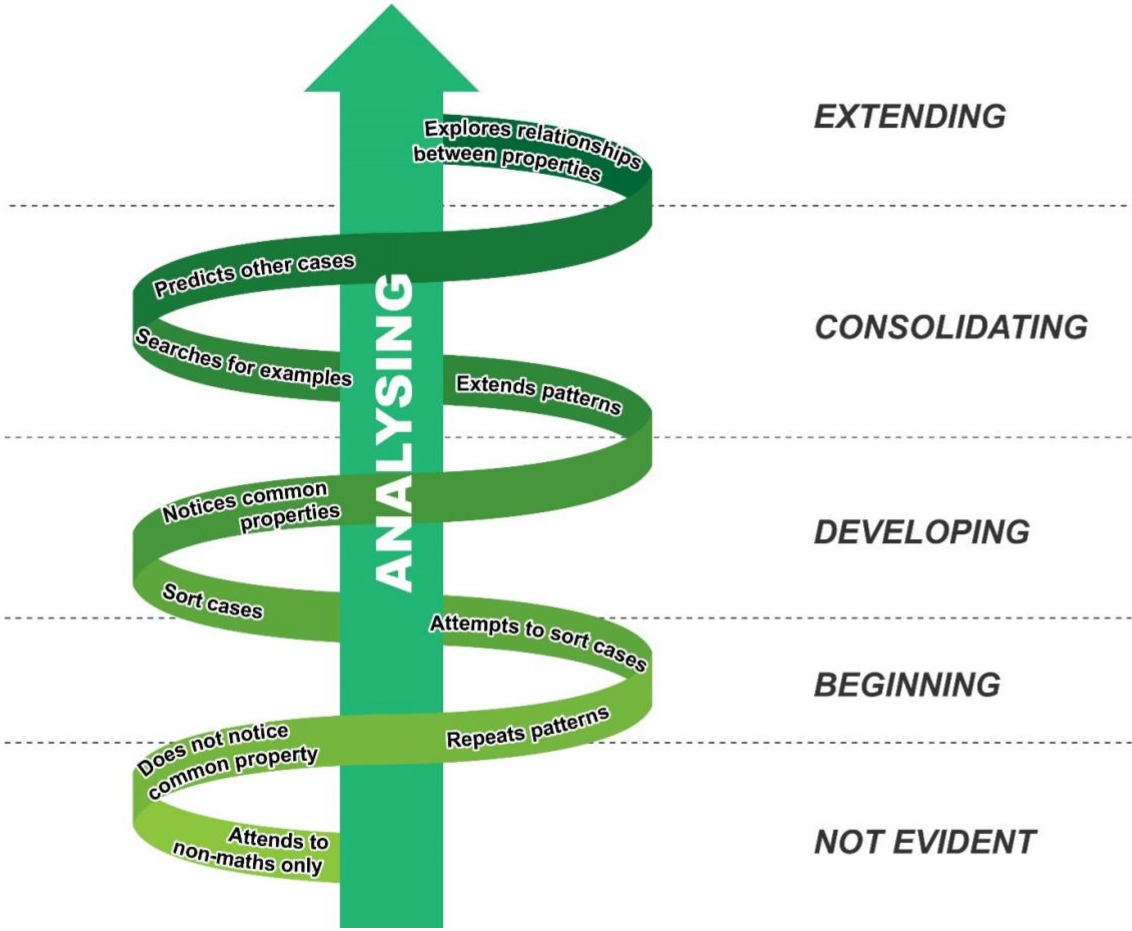
Alter an aspect of something to see (such and such) an effect. If we change this what will happen?

What follows from this? What do you think will happen next if we do this?

What do you notice?

When is it true?

Is it just sometimes true, or is it always true?



Generalising

Generalising involves ***forming conjectures*** (statements that are thought to be true but not yet known or shown to be true).

Generalising involves ***identifying common properties*** or patterns across more than one case and ***communicating a rule (conjecture)*** to describe the common property, pattern or relationship.

The ***statement or rule is communicated*** orally or written using words, diagrams or symbols.

The ***meaning of the statement or rule is communicated*** often using ***particular examples*** to explain the property or pattern. Further examples are used to explain how the rule applies to other cases, that is to show that how it is a generalisation.

As students develop, they communicate their conjectures and generalisations using more diverse and appropriate representations (language, diagrams and symbols) that reflect the mathematics concepts they have learned and are learning. They can communicate the common property, pattern or relationship that they noticed when analysing more precisely, explaining what it is and how it works more clearly.

The learning trajectory for generalising (shown opposite) represents the different levels of proficiency a student might display. Generalising that is higher up the trajectory will be evident in producing a conjecture that appears on current evidence to be correct, and is more clearly communicated and explained.

In order to develop proficiency in generalising, students need opportunities to explore patterns, search for common properties and be supported and challenged to form conjectures. Using reasoning prompts like those below will encourage students to generalise and more clearly explain their generalisations.

How can you describe what is the same?

What is the pattern here?

How can you describe the pattern?

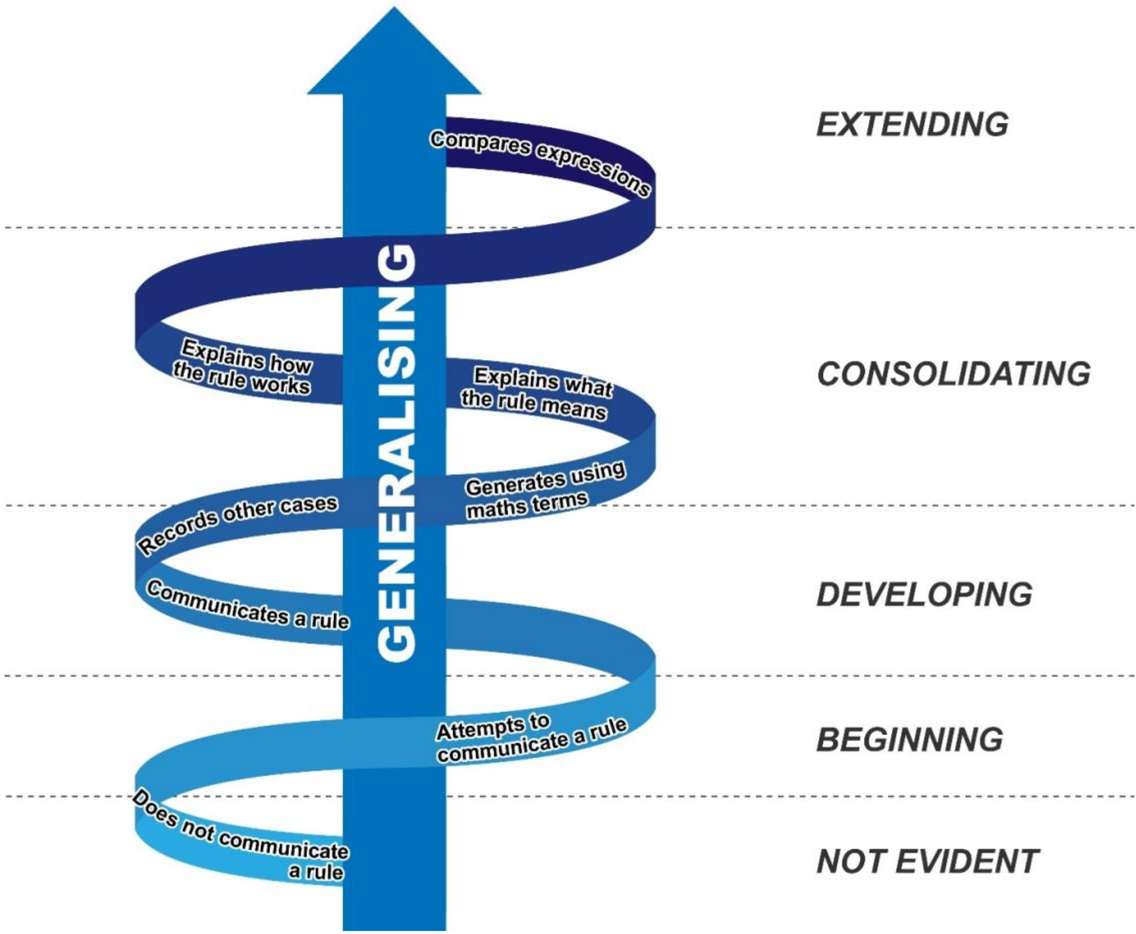
What happens in general?

What is the rule?

Is that … (pattern) always going to work?

Are there other examples that fit the rule?

How could you explain the rule to someone else?



Justifying

Justifying involves ***checking the truth of conjectures and generalisations*** to demonstrate or refute the truth of a claim.

Justifying uses ***logical argument*** to convince others of the truth of the claim or to refute the claim (demonstrate that the claim is false). A logical argument is convincing when it makes sense, is based on agreed facts, and leaves nothing unexplained.

A ***logical argument*** is made by:

- using ideas that are already understood;
- following agreed processes or steps for making arguments; and
- using terms, diagrams and symbols that are known and understood.

As students develop their capacity to justify, their logical arguments use more appropriate representations (language, diagrams and symbols), in accord with mathematics concepts they have learned and are learning.

While there are different types of logical argument, the justifying learning trajectory emphasises making an argument rather than relying on individual examples to verify a conjecture. An argument higher up in the trajectory offers deeper reasons, that are clearly logically connected, and covers more aspects of the conjecture.

Opportunities to justify can arise in almost any mathematics lesson. Reasoning prompts like those below engage students in justifying and encourage them to develop their arguments.

Is it (the conjecture) just sometimes true, or is it always true?

When is it (the conjecture) true?

How do you know?

How could we demonstrate/show/prove that it (the conjecture) is true?

True or false? Why? Let's justify.

Convince me.

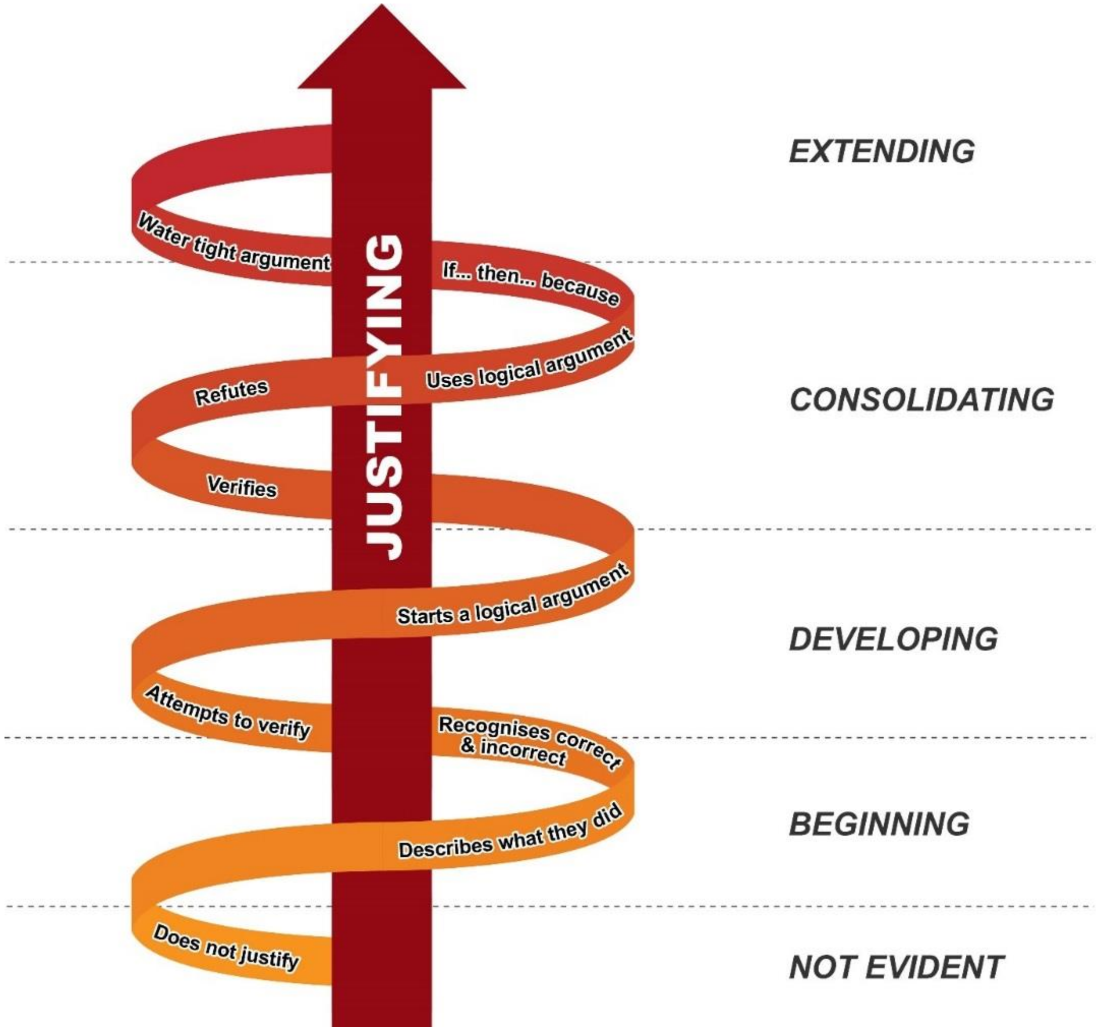
How can we be sure ...?

Tell me what is wrong with ...

Explain - why does this (process/rule/result) work?

Can you go through that step by step? Can you explain that step by step?

Why? *If...then...*



Sample assessment

A Magic V is a number puzzle. The aim is to arrange five consecutive numbers so that the sum of each 'arm' of the V is the same. Below are two Vs. The left V is a Magic V (4+2+3=9 and 5+1+3=9); the V on the right is not. Each number can be used only once.

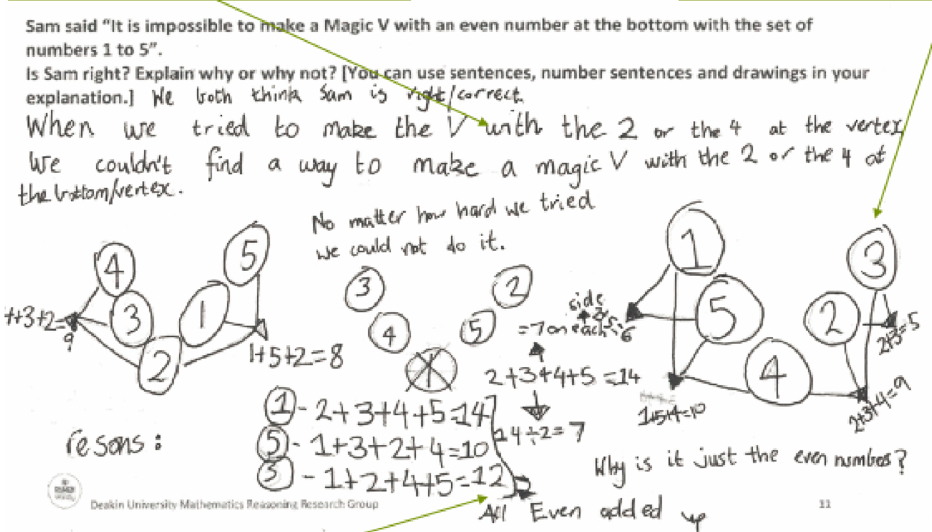
Sam said, "It is impossible to make a Magic V with an even number at the bottom with the set of numbers 1 to 5."

Is Sam right? Explain why or why not.

JUSTIFYING: Starting statements in a **logical argument** are correct and accepted by the class.

ANALYSING: Sorts and classifies according to a common property.

The student is analysing the total of the arms and sorting according to even and odd numbers at the vertex.



Sam said "It is impossible to make a Magic V with an even number at the bottom with the set of numbers 1 to 5".
Is Sam right? Explain why or why not? [You can use sentences, number sentences and drawings in your explanation.] We both think Sam is right/correct.
When we tried to make the V with the 2 or the 4 at the vertex we couldn't find a way to make a magic V with the 2 or the 4 at the bottom/vertex.
No matter how hard we tried we could not do it.

ANALYSING: Developing
GENERALISING: Beginning
JUSTIFYING: Developing

Teacher Prompt:

You have made a very interesting remark that if we have an even number at the bottom then the total of the two arms are not equal.

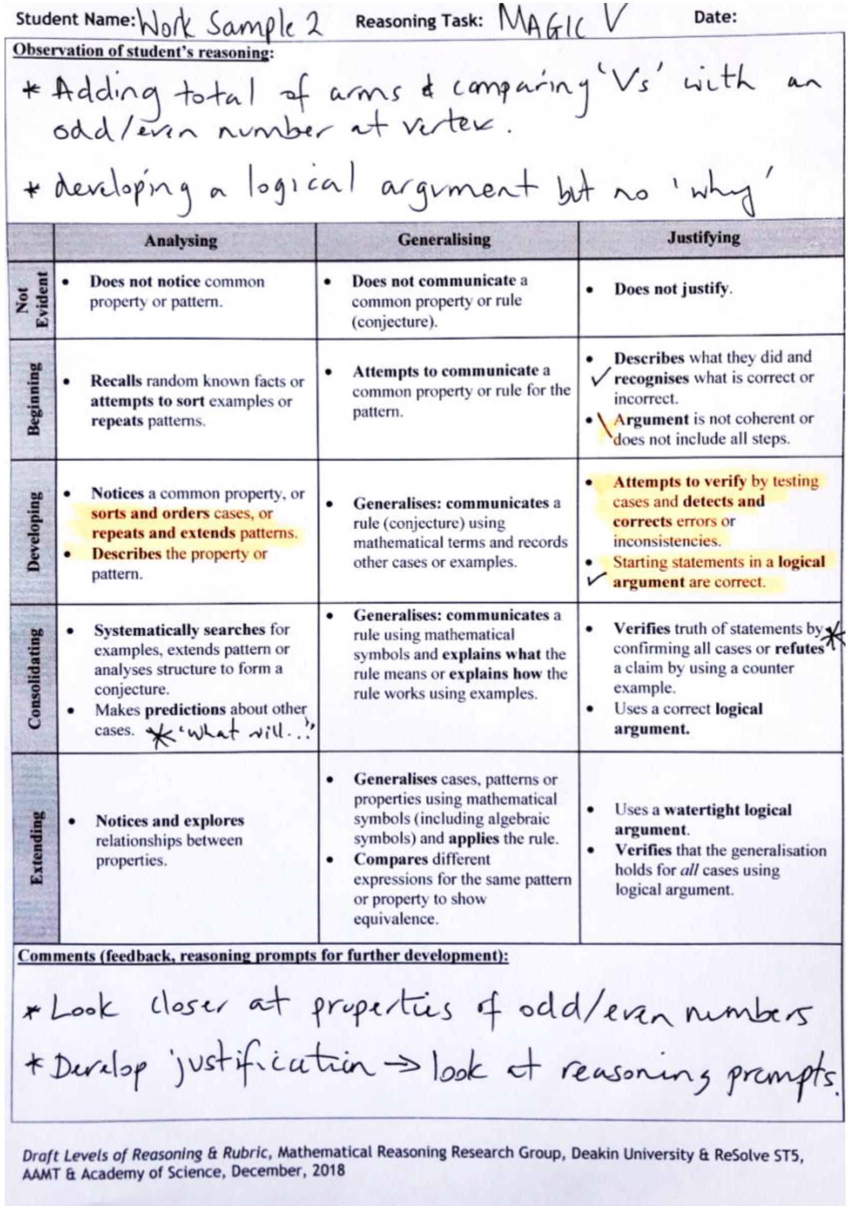
Is it true for all different Magic V with 2 and 4 at the bottom? Why?

What do we need to have at the bottom to make the two arms equal?

JUSTIFYING: Attempts to verify by testing cases or explaining the meaning of a conjecture using one example.

GENERALISING: Detecting and correcting errors and inconsistencies using materials, diagrams and informal written methods.

The next step is for the student to think about the properties of odd and even numbers.



Student Name: Work Sample 2 Reasoning Task: MAGIC V Date:

Observation of student's reasoning:

* Adding total of arms & comparing 'Vs' with an odd/even number at vertex.
* developing a logical argument but no 'why'

	Analysing	Generalising	Justifying
Not Evident	• **Does not notice** common property or pattern.	• **Does not communicate** a common property or rule (conjecture).	• **Does not justify**.
Beginning	• **Recalls** random known facts or **attempts to sort** examples or **repeats** patterns.	• **Attempts to communicate** a common property or rule for the pattern.	• **Describes** what they did and ✓ **recognises** what is correct or incorrect. • **Argument** is not coherent or does not include all steps.
Developing	• **Notices** a common property, or **sorts and orders** cases, or **repeats and extends** patterns. • **Describes** the property or pattern.	• **Generalises: communicates** a rule (conjecture) using mathematical terms and records other cases or examples.	• **Attempts to verify** by testing cases and **detects and corrects** errors or inconsistencies. • Starting statements in a **logical** ✓ **argument** are correct.
Consolidating	• **Systematically searches** for examples, extends pattern or analyses structure to form a conjecture. • Makes **predictions** about other cases. * 'what will..'	• **Generalises: communicates** a rule using mathematical symbols and **explains what** the rule means or **explains how** the rule works using examples.	• **Verifies** truth of statements by * confirming all cases or **refutes** a claim by using a counter example. • Uses a correct **logical argument**.
Extending	• **Notices and explores** relationships between properties.	• **Generalises** cases, patterns or properties using mathematical symbols (including algebraic symbols) and **applies** the rule. • **Compares** different expressions for the same pattern or property to show equivalence.	• Uses a **watertight logical argument**. • **Verifies** that the generalisation holds for *all* cases using logical argument.

Comments (feedback, reasoning prompts for further development):

* Look closer at properties of odd/even numbers
* Develop justification → look at reasoning prompts

Draft Levels of Reasoning & Rubric, Mathematical Reasoning Research Group, Deakin University & ReSolve ST5, AAMT & Academy of Science, December, 2018

Practical Advice: Rubric, Exemplars & Prompts

In order to assess a students' reasoning, we need to be able to see and/or hear their reasoning, and we need to know what to notice. The rubric outlines what to notice, and these ideas are illustrated in the exemplars provided with these resources.

Each exemplar includes student work samples at a variety of levels, and samples of teacher assessments, which link the student work to the rubric. A detailed list to help you select a suitable exemplar can be found here.

Before we can assess reasoning, we may need to encourage students to express their thinking so that we can see and/or hear it. These resources include reasoning prompts which have been designed to help students engage in reasoning and articulate their reasoning actions. These prompts are summarised on a classroom poster, and in a set of prompt cards which can be laminated. Other prompts are also included for enabling, extending, and facilitating discussion.

The formative assessment data gathered about the range of reasoning proficiency of the students in a class can then be used for planning lessons to enhance students' reasoning. This planning may involve selecting particular reasoning tasks that provide more opportunities for reasoning, or more consciously using prompts to elicit, support and challenge students' reasoning in all lessons.

Formative Assessment

These resources are designed to be used primarily as formative assessment that is, assessment **for** learning. This differs from summative assessment ('assessment **of** learning') which is used to measure student achievement to determine what standard they have achieved. Research has shown that formative assessment is very powerful.

> "Assessment ***for*** learning involves teachers using evidence about students' knowledge, understanding, and skills to inform their teaching. Sometimes referred to as 'formative assessment', it usually occurs throughout the teaching and learning process to clarify student learning and understanding.
>
> Assessment ***of*** learning assists teachers in using evidence of student learning to assess achievement against outcomes and standards. Sometimes referred to as 'summative assessment', it usually occurs at defined key points during a unit of work or at the end of a unit, term or semester, and may be used to rank or grade students. The effectiveness of assessment of learning for grading or ranking depends on the validity and reliability of activities. Its effectiveness as an opportunity for learning depends on the nature and quality of the feedback." (http://syllabus.nesa.nsw.edu.au/support-materials/assessment-for-as-and-of-learning/)

Advice on using the rubric

The rubric for assessment can be used:

- To assess individual students, or groups of students, or to plan learning for a whole class.
- During a lesson, or using written or recorded student work after a lesson.
- Alone, or collaboratively with colleagues.
- With the exemplar tasks, or with your own tasks.

The rubric, which summarises the various stages in the three learning trajectories, is presented in two ways. There is a full version (2 pages), and a summarised assessment sheet for class use.

The space at the top of the assessment sheet is to record a student's reasoning (oral, or communicated by gesture, or observed in written work and diagrams). You might then use a highlighter pen to select in the table the observed reasoning action. The space at the bottom of the sheet can be completed during the lesson or following the lesson: record prompts used during the lesson or that could be used in a future lesson and identify possible follow up or extension tasks in this space.

To use the rubric to assess a group of students, record the range of reasoning actions observed in the space at the top. You can then record the names of students in the relevant level (row) for each reasoning action (column). Use the space at the bottom to record possible future actions (prompts, follow up tasks).

Consider using the rubric with colleagues as a tool for planning classroom experiences. Teachers might work in pairs to analyse and discuss individual student's written work to reach agreement on the level of reasoning. Collaboration on assessment is helpful as it is not easy to interpret written records and consider all the possible meanings that students are attempting to communicate.

Some other tips:

- Separating reasoning into the three key reasoning actions gives a deeper understanding of reasoning. However, the boundaries between the three actions are somewhat blurred, so different people will justifiably allocate some features of student work to different reasoning actions. It is most important that teachers notice their students' reasoning, not that the assessment is watertight.
- Students communicate their reasoning through gesture, talking and writing using diagrams, words, and symbols. In order to notice and assess students' reasoning teachers need to watch and listen to their students as well as review what they write and record. Digital recordings that enable students to write, draw and verbally communicate their reasoning are useful.
- When students work in pairs, teachers can listen to students' reasoning. "Well, the kids were constantly having to explain, because they worked with partners which meant they could make their ***thinking out loud***, you could always **hear them *justifying***, thinking about other **reasons why** things won't work, or the reasons why things do work…" (Year 3/4 Primary school teacher).
- Communicating in writing is harder than communicating orally, but is a desirable skill for students to master.
- Students may not be at the same level for each of the three reasoning actions for a given task. For example, they may be considered 'consolidating' for analysing, 'developing' for generalising, and 'beginning' for justifying.
- The proficiency levels do not align with a school year level, though students in the higher year levels of primary school are expected to demonstrate reasoning at the upper end of the trajectory. Students in the early years may reason at a high level if the mathematical content of the task is of a suitable difficulty.
- Students' demonstrated proficiency may vary from task to task depending on the mathematical content of the task. This is because reasoning is intertwined with other proficiencies, especially understanding. For example, students with a strong number sense may be able to construct a logical argument about number, but if they are less familiar with geometry they may not know the language or geometric properties required to form a logical argument in this context. Assessment over a range of content is advised.

Advice on using the exemplars

The exemplars (see list) provide a sense of what mathematical reasoning learning experiences look like in the upper primary grades, and show how teachers can formatively assess the reasoning actions of their students.

The exemplars offer ways to *plan* tasks that embed the reasoning actions into everyday lessons: ways to *notice* mathematical reasoning when it happens and ways to *assess* student reasoning. The annotated work samples included in each exemplar provide the evidence that trial teachers used when assessing, illustrate the variety of levels of student reasoning, and show their assessment sheets.

The exemplars also provide short summaries of key reasoning actions at every level, as well as suggestions for reasoning prompts that might support and/or challenge students' reasoning.

Some other tips:

- Build the exemplar tasks into your normal teaching when the mathematical content is related.
- At many points during the lessons, for example when orienting students to the main task, you might like to follow a *think, pair, share* format.
- Working in pairs helps develop student's reasoning, because they can explain their thinking to each other.
- The exemplars include suggestions for follow up tasks that teachers might use to further develop and consolidate the reasoning actions that are the focus of that task.
- Each exemplar includes suggestions to show how the task may be adapted for different year levels, or gives tasks with a similar reasoning focus in a different format. The work samples provided with the exemplars are from Years 3, 4, 5 and 6 but the tasks can be adapted for both younger and older students.
- Teachers have noticed that some students surprise them and either do better than expected or not as well as expected on these tasks. This enables re-evaluation of the learning needs of the student.

Advice on using the prompts

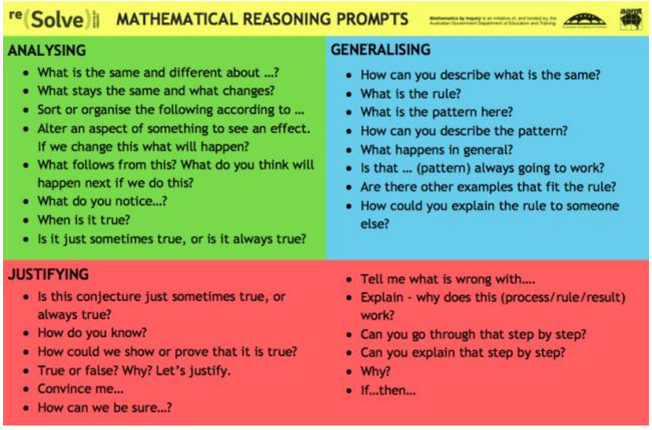
Students may need prompting to express their reasoning, and to think more carefully about their justifications. The exemplars give suggestions for prompts which will enable students to more fully engage with a task, to extend their thinking and to focus on reasoning. The intention is to provoke students to think more carefully and deeply rather than providing them with quick answers.

Reasoning Prompts (Cards or Poster)

Each exemplar includes questions to prompt students to reason, such as “How do you know…?”, “Explain why or why not” and “What if….?”. These prompts ensure that the focus of students’ learning is on their reasoning rather than just on correct answers. A poster of general prompts to encourage reasoning is provided for classroom display, along with a set of prompt cards.

Some other tips:

- Reasoning prompts can be used as students work, or when setting requirements for the task initially.
- When in doubt about what prompt to use, simply ask the student “Why?”
- Trial teachers reported that the laminated class set of cards was helpful in focusing students’ attention on the reasoning aspect of the task and hence supporting and/or extending their reasoning.



Enabling Prompts

An *enabling prompt* can help students who are experiencing difficulties with the main task to get started. The intention is that the students complete the enabling prompt then proceed with the main reasoning task.

Sullivan et al. (2006, pp. 502-503) suggest the following strategies to develop effective enabling prompts:

- Offering short focused explanations;
- Reducing the number of steps required for the task;
- Reducing the required number of variables;
- Simplifying the modes of representing results;
- Reducing the written elements in recording;
- Making the task more concrete;
- Reducing the size of numbers involved;
- Simplifying the language: or
- Reducing the physical demands of the manipulatives.

Enabling prompts include:

- *Providing a concrete example for students to analyse (Analysing)*
- *Prompting students to write down all that they know about the [number, shape, etc] (Analysing)*
- *Asking students to put into words the gestures that they are using to point out a pattern (Generalising)*
- *“Convince me that (the conjecture) works for all your cases/examples” (Justifying - verifying)*
- *Providing a hint about a case that would be a counter example: What if….? (Justifying - refuting)*
- *Providing the beginning stem for a logical argument: If…… (Justifying)*

Extending Prompts

Students finishing the main task quickly can be posed *extending prompts* to develop their thinking on an aspect of the main reasoning task. Sullivan et al. (2005, p. 124) have suggested the following for developing extending prompts:

- Offering an interesting exploration or puzzle that varies the context of the task;
- Proving the completeness of a set of answers; or
- Finding ways to generalise the result.

The exemplars include extending prompts such as repeating the main task with other numbers (Number Towers, Magic V, What Else Belongs?) or other shapes (Shapeshifter, Area and Perimeter). These extending prompts are appropriate for students who have demonstrated reasoning at a high level on the current task. Students who have completed the set task but have not demonstrated good reasoning should first be encouraged to do that.

Extending prompts for **generalising** include prompting students to find different rules that are also valid (so, different ways of expressing generality) (e.g. Matchsticks, Painted Cube) and to investigate relationships between common properties (e.g. What Else Belongs)? Extending prompts for analysing may point to finding further common properties and drawing in more detailed knowledge of mathematics connections. Extending prompts for **justifying** point to developing a tighter logical argument that more carefully fills in gaps in the reasoning, and covers more (hopefully even all) cases.

Discussion Prompts

Prompts for facilitating discussion (whole class or small group):

- *Prepare (perhaps in your group) to share your reasoning.*
- *Can you explain what [student] said?*
- *Who has a similar/different idea (or conjecture)? How is your idea (or conjecture) similar/different?*
- *What other examples fit with [student name's] conjecture?*
- *Who has a similar/different explanation/justification? How is your explanation similar/different?*

Interpreting students' work

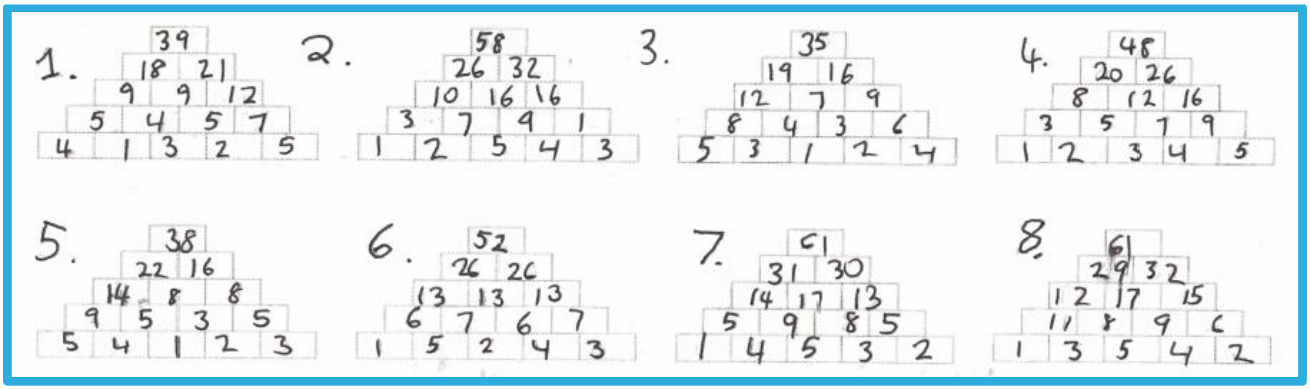
The following examples have been taken from the exemplars to illustrate each of the three key reasoning actions and to show how all aspects of a students' written work need to be considered (even small details). As is also evident in these reports, the teacher's observations from the classroom are of great assistance in interpretation.

Analysing

The following illustrates a student's analysis for the Number Towers problem (*Work Sample 3, Number Towers*).

In a Number Tower, each number is the sum of the two numbers below it in the tower. The task is to arrange the numbers 1 to 5 in the base row of the pyramid to produce the highest possible total at the top.

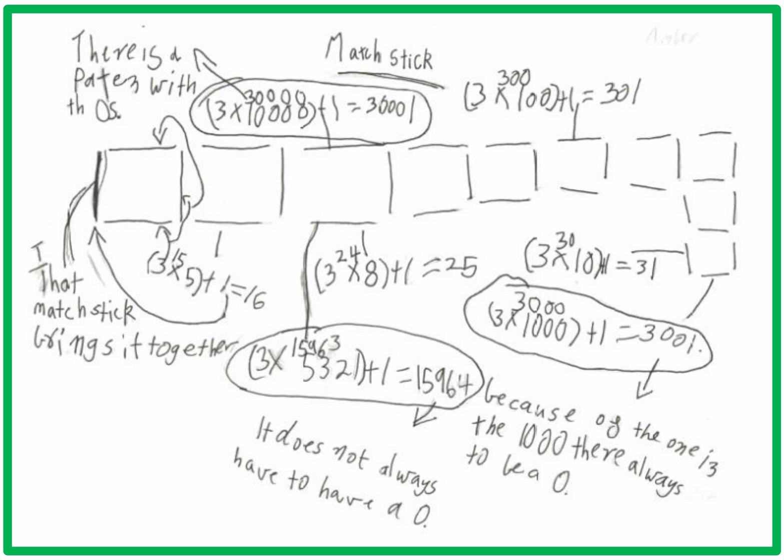
This student produced eight examples in the search for a rule. In the first five examples the student allocated the numbers randomly along the base. Then the student noticed a relationship between the placement of these numbers – big numbers near the centre make the total bigger. The student was then able to create the next three number towers so that the totals are ordered from lowest to highest. This student is noticing patterns, sorting cases and creating more cases to test ideas (Developing). An extending prompt, such as "What is the pattern here?" would assist this student to explain what they have noticed and to formulate a conjecture (i.e. generalise).



Generalising

This example demonstrates a student's generalising when solving the Matchsticks problem (*Work Sample 4, Matchsticks*) The student noticed a spatial pattern, which is revealed in the way the diagram shows the matchsticks in groups of 3, continuing to the right and down, and an initial match which "brings it together" on the extreme left. He also observed a numerical pattern for the total number of matches and illustrated this rule by calculating the number of matchsticks needed to make rows of 5, 8, 100, 1000 10000 and 5321 squares. The student also points out that the rule does not only apply to special lengths such as 100 and 100 (Extending).

In this work sample, the student communicates his rule by showing examples using mathematical symbols. He is likely to be on the cusp of communicating a general rule expressed in both words and symbols. An extending prompt such as “How could you write your rule so that it works for any number of squares?” would help him to take this step, reaching a statement such as “To work out the number of matchsticks needed to make a row of a certain length, multiply the length by 3 and then add 1.” Furthermore, near the rule for making the row of 5 squares, he drew the small arrows in the diagram to explain non-verbally *why* the rule works (i.e. 3 matchsticks added by extra square so multiply by 3, add 1). He can be encouraged to explain this explicitly.



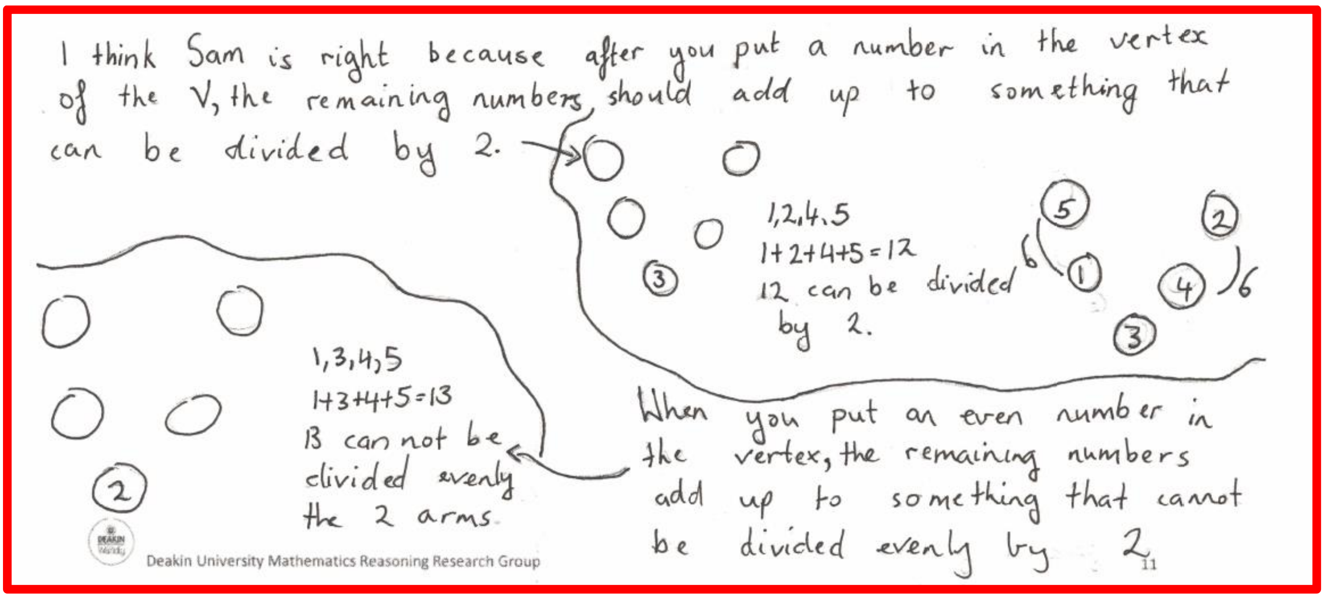
Justifying

A ‘Magic V’ is made by arranging the numbers 1 to 5 in a V shape, so that the totals along the two arms are equal. In the Magic V task students are asked to test the conjecture that the number at the vertex must be odd (so not 2 or 4). In the report below (*Work Sample 4, Magic V*), the student presents a logical argument to justify the conjecture (Extending). In constructing this argument she has used words, diagrams and mathematical calculations to explain why the conjecture is correct.

The first statement is “I think Sam is right because after you put a number in the vertex of the V, the remaining numbers should add up to something that can be divided by 2.” She justifies this statement by two examples, that are intended to convey generality even though they are completely specific. The drawing on the right shows how in a completed V, the two arms without the vertex nunber must be equal (arcs labelled with 6) and hence the sum of the remaining 4 numbers must be even. The drawing on the left illustrates with an example that if the total of the remaining 4 numbers is not even, then two equal-sum arms cannot be constructed.

Finally the student closes the argument with the observation: “When you put an even number in the vertex, the remaining numbers add up to something that cannot be divided evenly by 2.”

The only part of a water-tight argument that has not been explicitly mentioned is that because 1+2+3+4 +5 = 15 is odd, subtracting an even number (the vertex) will result in an odd total. So the vertex must be odd.



Leading Professional Learning

These resources are designed for use by individual teachers, and by groups of teachers with or without a formal leader. An ideal way to use them is as part of a whole school approach to emphasising the Australian Curriculum: Mathematics four proficiencies.

One recommended approach for a group is as follows. It requires about 2 hours of working together initially, and then later, a flexible 3 to 4 hours for small group assessment and whole school group meetings.

Step 1. Preparation

Read this Teachers' Guide thoroughly to familiarise yourself with the assessment rubric. Carefully examine several exemplars (including *What Else Belongs?* which is a very flexible task used in the slide show) to familiarise yourself with the tasks and the likely range of student responses.

Print or arrange digital copies for all participants of:

- the Reasoning Quick Guide (*ST5_Reasoning_QuickGuide.docx*) packaged with these resources;
- at least one exemplar of your choosing, including its work samples.

Step 2. Initial Professional Learning Session

Organise professional learning of about two hours in one or two sessions. Use the *Assessing Mathematical Reasoning* slide show *ST5_Reasoning _0a_ProfLearning.pptx* to discuss the nature of mathematical reasoning and its assessment.

The slide show has a brief audio accompanying each slide and presenter's notes for each slide. Some of the presenter's notes are extensive. The audio is designed especially for teachers working individually through the resources, but may also be useful in a group presentation.

In the session, teachers should

- See and discuss descriptions and examples of mathematical reasoning
- Observe how a lesson can have its mathematical purpose focussed on the reasoning proficiency not exclusively on content descriptors
- Review questioning strategies that support and challenge students' reasoning
- Learn about the three key reasoning actions
- See the use of assessment rubric and the assessment sheet
- Make plans to try some exemplar tasks with classes

Step 3. Gathering student work

Student work is collected.

Teachers use selected exemplar tasks or adaptations for their own year levels, and trial the use of the assessment sheet and reasoning prompts. The sheet can be used in class to record observations and with student written work.

Step 4. Working together to assess and plan.

Teachers work together to discuss the assessment of the reasoning evident in the students' work. This might first be done in year level groups, or by pairs of teachers, with everyone coming together later to report. Teachers also plan how to use the information gathered as formative assessment to improve reasoning across the school.

Selecting an Exemplar: Useful Information

Exemplar	Main Year Level (Adaptable Year Levels)	Reasoning Focus and Mathematics Content	Australian Curriculum Links	Description
Is it True?	**3** (2, 4, 5)	**Reasoning focus:** • Analysing: Exploring and noticing relationships between numerical structures • Justifying: Verifying or refuting statements as true **Content focus:** • Place Value • Additive Thinking	**ACMNA053:** Apply place value to partition, rearrange and regroup numbers to at least 10 000 to assist calculations and solve problems • justifying choices about partitioning and regrouping numbers in terms of their usefulness for particular calculations	Is it True? presents students with an inaccurate addition problem and asks them to explain and justify why this statement is true or not true. The content focus of the task is on place value, and students' understanding of grouping, re-grouping and renaming for one-digit, two-digit, and three-digit addition. While it is targeted at Year 3 students, the task can be extended to upper primary students with the use of decimals.
Number Towers	**3** (4, 5, 6)	**Reasoning focus:** • Analysing: Exploring and noticing relationships between numerical structures • Generalising: Comparing and contrasting to form conjectures • Justifying: Testing conjecture using examples to verify **Content focus:** • Additive Thinking	**ACMNA054:** Recognise and explain the connection between addition and subtraction **ACMNA055:** Recall addition facts for single-digit numbers and related subtraction facts to develop increasingly efficient mental strategies for computation	Number Towers gives students an opportunity to develop and test conjectures and form generalisations by reasoning mathematically about numerical structures with addition. The task promotes careful analysis of a mathematical structure. Students need to experiment systematically, keep track of results, and choose cases carefully to test the rule. The task can be adapted for older students by using fractions or decimals.
Magic V	**4** (3, 5)	**Reasoning focus:** • Analysing: Comparing and contrasting to notice a common property • Justifying: Constructing a logical argument based on properties of even and odd numbers **Content focus:** • Properties of Number	**ACMNA071:** Investigate and use the properties of odd and even numbers	The Magic V is a task which explores mathematical reasoning and affords students an opportunity to develop and test conjectures and form generalisations. By manipulating numbers and recording their ideas, students will have the opportunity to explore properties of odd and even numbers. The main purpose of this task, however, is to provide opportunities for students to develop their reasoning capacity.

Exemplar	**Main Year Level** (Adaptable Year Levels)	Reasoning Focus and Mathematics Content	Australian Curriculum Links	Description
Matchsticks	**4** (3, 5, 6)	**Reasoning focus:** • Analysing: Noticing pattern in number and space • Generalising: Communicating a conjecture about the pattern using words or symbols • Justifying: Explaining why the rule works **Content focus:** • Algebraic Thinking	**ACMNA081:** Explore and describe number patterns resulting from performing multiplication • identifying examples of number patterns in everyday life	Students make rows of squares from match sticks, and explore patterns than can be described pictorially, numerically and symbolically. Students learn to work systematically by keeping a record of results that encourage them to develop and test conjectures and to ask themselves questions about finding further cases. As the students describe and explain patterns, they will begin to move from additive to multiplicative reasoning.
What Else Belongs?	**5** (1, 2, 3, 4, 6)	**Reasoning focus:** • Analysing: Noticing common properties of numbers • Generalising: Writing a statement (conjecture) about the common property • Justifying: Verifying that each number belongs **Number focus:** • Multiplicative Thinking • Number sense	**ACMNA098:** Identify and describe factors and multiples of whole numbers and use them to solve problems • exploring factors and multiples using number sequences • using simple divisibility tests	What Else Belongs? is a task in which students find at least two numbers which belong to the set of given numbers. They notice and describe properties of number; such as size, order, composition, place value, multiples, factors, even or odd. The students justify why some numbers have similar or dissimilar properties. This task has been written for Year 5 students, however, it can be used for Year 3 and 4 students and extended for Year 6 students by including a set of numbers that include decimals, common fractions, mixed numbers, and improper fractions.
Shapeshifter	**4** **(5)**	**Reasoning focus:** • Analysing: Conducting trials to form a conjecture • Generalising: Writing and illustrating a rule • Justifying: Explaining why the rule works for all polygons. **Content focus:** • Polygons	ACMMG088: Compare and describe two dimensional shapes that result from combining and splitting common shapes, with and without the use of digital technologies	Shapeshifter is based on the picture story book "The Greedy Triangle." For this task the students are the "shapeshifter' in the story and find a rule for using a straight line to dissect a polygon (2 dimensional shape) to make a polygon with one more side.

Exemplar	**Main Year Level** (Adaptable Year Levels)	Reasoning Focus and Mathematics Content	Australian Curriculum Links	Description
Area and Perimeter	**6** **(4, 5)**	**Reasoning focus:** • Analysing: Producing examples to test conjecture • Justifying: Using a counter example to refute a statement **Content focus:** • Measurement • Multiplicative Thinking	**ACMMG137:** Solve problems involving the comparison of lengths and areas using appropriate units • recognising and investigating familiar objects using concrete materials and digital technologies	Area and Perimeter will engage students in responding to a conjecture in relation to a common misconception of the link between the perimeter and area of rectangles: that a larger perimeter will always result in a larger area. Students will learn that it is sufficient to offer one counter example to refute a conjecture or general statement for all cases.
Painted Cube	**6** (5)	**Reasoning focus:** • Analysing: Analyses structures to form a conjecture. • Generalising: Communicates the rule for any case using words or symbols, including algebraic symbols. • Justifying: Uses a logical argument that is mathematically sound **Content focus:** • Algebraic Thinking • 3 dimensional geometric shapes	**ACMNA176:** Create algebraic expressions and evaluate them by substituting a given value for each variable • using authentic formulas to perform substitutions	The Painted Cube task is rich and complex, providing students with opportunities to explore a variety of patterns that can be described spatially, numerically and symbolically. Students learn to work systematically by keeping a clear record of results which will encourage them to develop and test conjectures and to ask themselves questions about further cases.

More on Reasoning

Reasoning is the mathematical thinking that creates and validates mathematical ideas and new knowledge. It is meaning making. Students discover and make sense of mathematical ideas and concepts when engaged in inquiry-based tasks that require them to reason. Reasoning tasks in mathematics provide opportunities for creative thinking. Hence reasoning, one of four proficiencies in the Australian Curriculum: Mathematics is intertwined with the other proficiencies: understanding, fluency and problem-solving.

Reasoning in the Australian Curriculum: Mathematics is defined as:

> Students develop an increasingly sophisticated capacity for logical thought and actions, such as **analysing**, **proving**, **evaluating**, **explaining**, **inferring**, **justifying** and **generalising**. Students are reasoning mathematically when they **explain** their thinking, when they **deduce** and **justify** strategies used and conclusions reached, when they **adapt** the known to the unknown, when they **transfer** learning from one context to another, when they **prove** that something is true or false and when they **compare** and **contrast** related ideas and **explain** their choices.

The key idea here is logical thought. This involves being convincing and providing evidence, just as is required in argumentative writing. As well as clearly *explaining how* a process is carried out, a logical argument must *explain why* something occurs or a process works. In ordinary conversation, it is not sufficient to justify why something is true by explaining the process of doing it or giving a few examples of instances where it is true; rather it is necessary to make a logical argument by drawing together known facts to reach a conclusion. It is the same in mathematics.

Types of logical argument include:

- verifying and validating the claim by showing how the rule works for all cases (exhaustion),
- refuting a claim by providing a counter example,
- refuting a claim by showing a contradiction
- deduction (i.e. using one or more logical steps arguing from a premise to a conclusion if… then … hence….)
- using general representation of the claim along with logical steps to show how the claim applies to all cases (generic proof)

Reasoning is important to both the understanding proficiency and the problem solving proficiency. There are many reasoning actions listed in the Australian Curriculum definition above, which contribute to creating new knowledge and understanding for the learner. Students who see how mathematical steps are logically connected, and are able to 'fill in gaps' by generalising, and can explain and justify what they have to learn, make sense of mathematics.

Mathematical reasoning is also an important component at every step of the problem solving process. The distinction can be clarified by comparing the verbs in the description of reasoning with the verbs included in the problem solving statement from the Australian Curriculum as shown in the following table. For problem solving, students need to make sense of the problem by interpreting it and then make choices about how to represent it or model it mathematically and decide how to solve it. Communicating effectively involves explaining the process for solving the problem as well as showing that the solution makes sense for the problem. Reasoning is involved within all of these aspects of problem solving, just as it is involved in the understanding proficiency whenever we teach emphasising connections and sense making. It is essential that students bring a capacity of logical thought and reasoning to all their learning.

Reasoning	**Problem solving**
Students develop capacity for logical thought and actions: • analysing • proving • evaluating • explaining • inferring • justifying • generalising	Students develop the ability to: • make choices • interpret • formulate, model and investigate problem situations • communicate solutions effectively

Actions and capabilities for the reasoning and problem solving proficiencies (ACARA, 2017)

Key Reasoning Actions

These Assessing Reasoning resources focus on three key reasoning actions: Analysing, Generalising, and Justifying. These three reasoning actions incorporate the other verbs describing reasoning that are included in the Australian Curriculum: Mathematics as shown below. For example, proving and evaluating are both forms of justifying, and inferring is required to generalise.

These resources focus on three key reasoning actions: *Analysing, Generalising,* and *Justifying*. These have been chosen because they cover a range of other actions, for example justifying may involve proving and explaining.

Analysing	**Generalising**	**Justifying**
Comparing and contrasting Exploring	Inferring Transferring Explaining (what) Adapting	Logical thought Explaining (why) Proving Evaluating Deducing

Key reasoning actions in the Australian Curriculum: Mathematics (ACARA, 2017)

Students at all year levels in primary school can analyse, generalise and justify for each mathematics content domain. When analysing, generalising and justifying, students draw on what they have learned and how to represent it to discover something new or make sense of something.

> "They're bringing in all of their knowledge and they're putting it all together and figuring bigger things out through reasoning." (Year 3/4 Primary school teacher).

Key Reasoning Actions: Distinct but Working Together

Problems that provide strong opportunity for students to reason or that have learning goals for students to reason will require students to use at least one of these key reasoning actions. Some mathematics tasks will require students to use all three reasoning actions, such as the exemplars 'What Else Belongs?' and 'Number Tower'. For Number Tower, students need to analyse in order to see the effect of bottom position on the top number, form a conjecture (that is, state a generalisation that it is always best to have the largest number in the centre) and then they need to test their conjecture, and justify it. Other tasks for example 'Is it True?' and "Magic V' provide students with a conjecture that is thought to be true but not known or shown to be true. Students are required to test the conjecture or justify the statement and so will mainly use analysing and justifying. Of course, while they are investigating the statement, they may undertake some mini-cycles involving generalising as well.

The three reasoning actions are intended to cover the activity of mathematical reasoning in a simple way, giving some insight into the strengths of students' reasoning and where they need further learning. However, it is not always possible to easily draw boundaries between the three actions, so that distinctions between them are sometimes blurred. Different teachers may allocate particular actions differently. When this happens, try to look beyond the labels to see the student's strengths and also look to where they may need further assistance.

Further Information

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Recommended websites

Below is a list of suggested websites offering an abundance of free resources that promote mathematical reasoning. The resources in these sites offer lesson suggestions that teachers can use in conjunction with the Mathematical Reasoning Rubric to support their everyday planning, teaching and assessment of students' mathematical reasoning.

Organisation	Description	Link
reSolve	The reSolve project is an Australian national initiative that provides detailed materials to support teachers in facilitating mathematical inquiries in their classrooms.	http://resolve.edu.au/
Top Drawer	Australian Association of Mathematics Teachers (AAMT). Contains descriptions of the 'Big Ideas' for reasoning and reasoning-specific tasks along with other recommendations to support teachers in teaching with a reasoning focus in mind.	https://topdrawer.aamt.edu.au/Reasoning
ACARA	Australian Curriculum Assessment and Reporting Authority. This resource is a recent addition to the Australian Curriculum website that contains work samples demonstrating student reasoning.	https://www.australiancurriculum.edu.au/resources/mathematics-proficiencies/
TTML	Task Types and Mathematical Learning is a set of classroom activities designed by Victorian classroom teachers as part of a research project conducted by Monash University and the Australian Catholic University.	http://www.education.vic.gov.au/school/teachers/teachingresources/discipline/maths/Pages/ttml.aspx
Mathematics Task Centre	Black Douglas Professional Education Services. Contains tasks which encourage students to work like mathematicians.	http://mathematicscentre.com/taskcentre/
NRICH (UK)	NRICH: Enriching mathematics. Reasoning: Identifying Opportunities (Article)	https://nrich.maths.org/109 90
NRICH (UK)	Search for tasks by topic (content). Each lesson contains lesson descriptions for teachers, sample student solutions and additional classroom support materials.	https://nrich.maths.org/public/leg.php
NRICH (UK)	266 lessons which foster mathematical reasoning. This list can be sorted by year levels.	https://nrich.maths.org/public/leg.php?code=71
NZ Maths	Lessons and tasks according to content strands of the New Zealand curriculum	https://nzmaths.co.nz/
NCTM (USA)	National Council of Teachers of Mathematics (NCTM). Standards-based lessons and resources, including interactive tools for students.	http://illuminations.nctm.org/Lessons-Activities.aspx
NCTM (USA)	Activities with Rigor and Coherence	https://www.nctm.org/ARC s/

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Authors

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Sources of Exemplar Tasks

Some exemplar tasks are adapted from other sources. Some tasks have long histories, but only the direct source is credited here.

Is it True? The main task has been adapted from work created by Dr Leicha Bragg in conjunction with staff from Lynnmour Elementary, North Vancouver, Canada. The Bakery task has been adapted from the ReSolve website: http://resolve.edu.au/year-5-multiplication-resolve-bakery?fromexploreresources=true

Number Towers. Adapted from 'Number Tower' task developed by the Australian Mathematical Sciences Institute.

Magic V. Adapted from NRICH Maths, Cambridge University. See https://nrich.maths.org/6274 , https://nrich.maths.org/7821 and https://nrich.maths.org/1835

Matchsticks. Adapted from http://www.transum.org/Maths/Activity/Matchstick_Patterns/

What Else Belongs? Adapted from Marion Small (2011). One, two, infinity website: http://www.onetwoinfinity.ca/

Painted Cube. Adapted from the 'Painted Cube' task presented by the Black Douglas Professional Education Services http://mathematicscentre.com/taskcentre/160paint.htm and the work of Dr Leicha Bragg and Dr Heather Kelleher, University of British Columbia, Canada.

Student Name: **Reasoning Task:** **Date:**

Observation of student's reasoning:

Comments (feedback, reasoning prompts for further development):

	ANALYSING	GENERALISING	JUSTIFYING
NOT EVIDENT	• **Does not notice** common property or pattern.	• **Does not communicate** a common property or rule (conjecture) for a pattern.	• **Does not justify**.
BEGINNING	• **Recalls** random known facts or **attempts to sort** examples or **repeats** patterns.	• **Attempts to communicate** a common property or rule (conjecture) for a pattern.	• **Describes** what they did and **recognises** what is correct or incorrect. • A**rgument** is not coherent or does not include all steps.
DEVELOPING	• **Notices** a common property, or **sorts and orders** cases, or **repeats and extends** patterns. • **Describes** the property or pattern.	• **Generalises: communicates** a rule (conjecture) using mathematical terms, and records other cases or examples.	• **Attempts to verify** by testing cases, and **detects and corrects** errors or inconsistencies. • Starting statements in a **logical argument** are correct.
CONSOLIDATING	• **Systematically searches** for examples, extends patterns, or analyses structures, to form a conjecture. • Makes **predictions** about other cases.	• **Generalises: communicates** a rule (conjecture) using mathematical symbols and **explains what** the rule means or **explains how** the rule works using examples.	• **Verifies** truth of statements by confirming all cases or **refutes** a claim by using a counter example. • Uses a correct **logical argument.**
EXTENDING	• **Notices and explores** relationships between properties.	• **Generalises** cases, patterns or properties using mathematical symbols and **applies** the rule. • **Compares** different expressions for the same pattern or property to show equivalence.	• Uses a **watertight logical argument**. • **Verifies** that the generalisation holds for *all* cases using logical argument.

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Assessing Mathematical Reasoning: Rubric

This table elaborates the different levels of proficiency for each key reasoning action. It provides more detail than the classroom assessment sheet, and should be used as a guide when looking at student work.

The levels reflect the learning trajectories. There are several descriptors for each level, for each key reasoning action, but not all descriptors need to be present for a student to be at that level. One reason is that different tasks bring out different reasoning actions. For example: *What Else Belongs?* requires verification and logical argument whereas *Area and Perimeter* requires refutation using a counter example (although there are opportunities for other reasoning in the follow up tasks). The exemplars provide examples of students' reasoning matched to descriptors in this rubric.

	ANALYSING	GENERALISING	JUSTIFYING
NOT EVIDENT	• **Does not notice** numerical or spatial structure of examples or cases. • Attends to **non-mathematical** aspects of the examples or cases.	• **Does not communicate** a common property or rule for a pattern.	• **Does not justify**. • **Appeals** to teacher or others.
BEGINNING	• **Notices** similarities across examples • **Recalls** random known facts related to the examples. • Recalls and **repeats patterns** displayed visually or through use of materials. • Attempts to **sort** cases based on a common property.	• **Draws attention to** or **attempts to communicate** a common property or repeated components of a pattern using: ○ body language (gesture), ○ drawing, ○ concrete materials ○ counting or • oral language (metaphors).	• **Describes** what they did and why it may or may not be correct. • **Recognises** what is correct or incorrect using materials, objects, or words. • Makes **judgements** based on simple criteria such as known facts. • The **argument may not be coherent** or include all steps in the reasoning process.
DEVELOPING	• **Notices** a common numerical or spatial property. • **Recalls** and **repeats** patterns using numerical structure or spatial structure. • **Sorts and classifies** cases according to a common property. • **Orders** cases to show what is the same or stays the same and what is different or changes. • **Describes** the case or pattern by labelling the category or sequence.	• **Communicates** a rule (conjecture) about a: ○ *property* using words, diagrams or number sentences. ○ *pattern* using words, diagrams to show recursion or number sentences to communicate the pattern as repeated addition. • **Records** other cases that fits the rule (conjecture) or extends the pattern using the rule.	• **Attempts to verify** by testing cases or explaining the meaning of a conjecture using one example. • **Detecting and correcting errors** and inconsistencies using materials, diagrams and informal written methods. • Starting statements in a **logical argument** are correct and accepted by the classroom.

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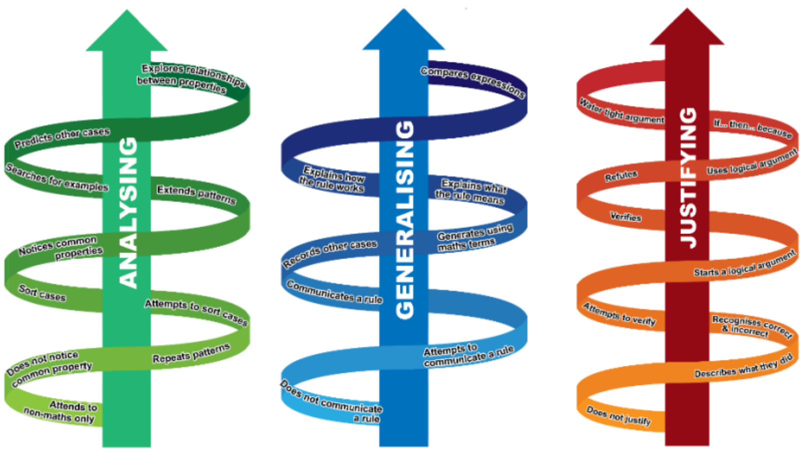
CONSOLIDATING	• **Notices more than one** common property by systematically generating further cases and/or listing and considering a range of known facts or properties.
	• **Repeats and extends patterns** using both the numerical and
	spatial structure.
	• **Searches for and produces examples**:
	  ○ using tools, technology and modelling
	• Makes **predictions** about other cases:
	  ○ with the same property
	  ○ included in the pattern
		• **Generalises: communicates a rule (conjecture)** using mathematical terms, symbols or diagrams (e.g. a number sentence or labelled geometric diagram)
		• **Explains what** the rule (conjecture) means using one example.
		• **Extends** the pattern using an example to **explain how** the rule works.
			• **Verifies** truth of statements by using a common property, rule or known facts that confirms each case. May use materials and informal methods.
			• **Refutes** a claim by using a counter example.
			• Uses a correct **logical argument** that has a complete chain of reasoning and uses words such as 'because', 'if…then…', 'therefore', 'and so', 'that leads to'
			• **Extends the generalisation** using logical argument.

EXTENDING	• **Notices and explores** relationships between:
	  ○ common properties
	  ○ numerical structures of patterns.
		• **Generalises: communicates** the rule (conjecture) using mathematical symbols.
		• **Applies** the rule to find further examples or cases.
		• **Generalises** properties by forming a statement about the relationship between common properties.
		• **Compares** different symbolic expressions used to define the same pattern to show equivalence.
			• Uses a **watertight logical argument** that is mathematically sound and leaves nothing unexplained.
			• **Verifies** that the statement is true or the generalisation holds for *all* cases using logical argument.



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MATHEMATICAL REASONING PROMPTS

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ANALYSING

- What is the same and different about …?
- What stays the same and what changes?
- Sort or organise the following according to …
- Alter an aspect of something to see an effect. If we change this what will happen?
- What follows from this? What do you think will happen next if we do this?
- What do you notice…?
- When is it true?
- Is it just sometimes true, or is it always true?

GENERALISING

- How can you describe what is the same?
- What is the rule?
- What is the pattern here?
- How can you describe the pattern?
- What happens in general?
- Is that … (pattern) always going to work?
- Are there other examples that fit the rule?
- How could you explain the rule to someone else?

JUSTIFYING

- Is this conjecture just sometimes true, or always true?
- How do you know?
- How could we show or prove that it is true?
- True or false? Why? Let's justify.
- Convince me…
- How can we be sure…?
- Tell me what is wrong with….
- Explain - why does this (process/rule/result) work?
- Can you go through that step by step?
- Can you explain that step by step?
- Why?
- If…then…

Analysing Prompt Cards

What is the same and different about …?	What stays the same and what changes?
Sort or organise the following according to …	Alter an aspect of something to see an effect: If we change this what will happen?
What follows from this? What do you think will happen next if we do this?	What do you notice…?
When is it true?	Is it just sometimes true, or is it always true?

Generalising Prompt Cards

How can you describe what is the same?	What is the rule?
What is the pattern here?	How can you describe the pattern?
What happens in general?	Is that…(pattern) always going to work?
Are there other examples that fit the rule?	How could you explain the rule to someone else?

Justifying Prompt Cards (1)

Is this conjecture just sometimes true, or always true?	How do you know?
How could we show or prove that it is true?	True or false? Why? Let's justify.
Convince me…	How can we be sure?
Tell me what is wrong with…	Explain - why does this work?

Justifying Prompt Cards (2)

Can you go through that step by step?	Can you explain that step by step?
Why?	If…then…