Year 10/10A: Proof
Lesson 6: Peaucellier's Linkage

Australian Curriculum: Mathematics (Year 10 and 10A)

ACMMG243: Formulate proofs involving congruent triangles and angle properties.

- Applying an understanding of relationships to deduce properties of geometric figures (for example the base angles of an isosceles triangle are equal).

ACMMG244: Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes.

- Performing a sequence of steps to determine an unknown angle giving a justification in moving from one step to the next.
- Communicating a proof using a sequence of logically connected statements.

ACMMG272: Prove and apply angle and chord properties of circles

- performing a sequence of steps to determine an unknown angle or length in a diagram involving a circle, or circles, giving a justification in moving from one step to the next

Lesson abstract

Peaucellier's linkage converts circular motion to linear motion. Students explore how the linkage moves by constructing a physical model and using a computer simulation of the linkage. They discover how the geometric design of the linkage allows it to produce exact linear motion.

Mathematical purpose (for students)

To prove that Peaucellier's linkage converts circular motion to linear motion.

Mathematical purpose (for teachers)

This lesson provides a challenging task for students who can already work competently with angle chasing and similar triangles and are familiar with angle properties associated with circles. The geometric proof involves knowledge of rhombus properties, congruent and similar triangles, angle in a semicircle and Pythagoras' theorem.

Lesson Length 120 minutes approximately

Vocabulary Encountered

- collinear

Lesson Materials

- Per pair of students: 6 short blue and 2 long yellow Geo Strips, 6 paper fasteners, piece of A3 card.
- Slide show: *ST1_Yr10A_6a_PeaucellierImages.pptx*
- GeoGebra file *Peaucellier.ggb* or GeoGebra Tube link https://ggbm.at/JEe5a6KY
- GeoGebra file *PeaucellierProof.ggb* or GeoGebra Tube link https://ggbm.at/TAUVG9Rq
- GeoGebra file Peaucellier's Linkage Proof
- Student Sheet 1 – Peaucellier's Linkage (1 per student)
- YouTube video (optional) https://youtu.be/hSdW-i3nO1M
- YouTube video (optional) https://youtu.be/XotLCt4bBlM

We value your feedback after these lessons via https://www.surveymonkey.com/r/2JH6Z82

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Introducing Peaucellier's Linkage

The flourishing of mechanical invention that occurred as a result of the industrial revolution in the 18th and 19th centuries led to the development of a number of linkages for converting circular motion into linear motion and vice versa. Steam engines began to power transport and factories. Up and down motion could be produced by pistons, whereas circular motion (rotating wheels) was needed for trains and in factories (e.g. for conveyor belts, or drive shafts to transmit power around the factory). Machines in the factory (such as sewing machines) had to convert this circular motion back to up and down linear motion, and many linkages and other devices were designed to do this.

Some other linkages were designed instead to provide support to keep part of a machine moving in a straight line. Linkages were able to do this with minimal friction and kept the machines efficient. Some linkages, for example those of Watt and Chebycheff, produced *approximately* linear motion over part of the motion, whereas others, including Peaucellier's linkage, produced exact linear motion. Simple three-bar or four-bar linkages which produced approximate linear motion had the advantage of less moving parts, and hence less wear, whereas the linkages which produced true linear motion were less practical, and have survived only as mathematical curiosities.

Peaucellier (pronounced Po/sell/ee/ay) was a French engineer who invented the linkage in 1864. It was the first linkage to be invented that converted circular motion to geometrically true linear motion. The linkage is also referred to as the Peaucellier-Lipkin linkage as it was invented independently by Lipkin and he first published details of the linkage.

Getting started

Show students the initial images from the slide show *ST1_Yr10A_6a_PeaucellierImages.pptx* and a video of Peaucellier's linkage produced by Chris Sangwin available at https://youtu.be/hSdW-i3nO1M

"Straight Line Mechanisms" https://youtu.be/XotLCt4bBlM by Kevin Stamp–shows many linkages including those of Watt, Chebycheff and Peaucellier.

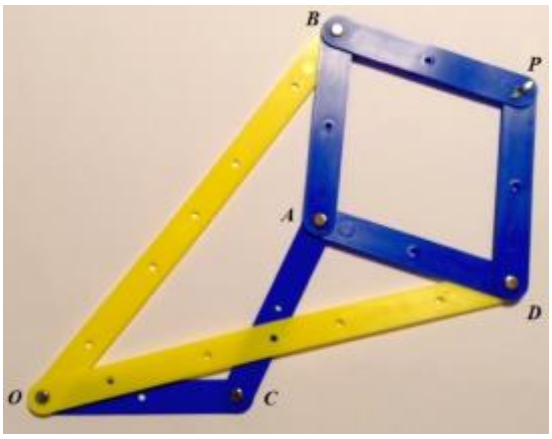
Models of Peaucellier's Linkage

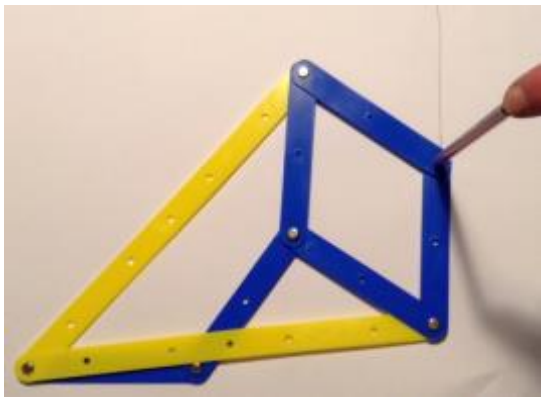
Physical model

Peaucellier's linkage consists of eight bars of fixed length. Points *O* and *C* are fixed (the paper fasteners are inserted through the piece of A3 card) and the lengths of *OC* and *CA* are equal. The lengths *OB* and *OD* are equal. The strips *AB*, *BP*, *PD*, and *DA* are all equal in length, forming a rhombus. In this model, the same length blue strips have been used for *OC* and *CA* as for *AB*, *BP*, *PD*, and *DA*, but a different length strip could have been used for *OC* and *CA* (although those two must be equal).

Students should observe the features of the model and the motion, noting for example, that:

- *ABPD* is a rhombus.
- Triangle *ABD* is isosceles.
- Point *A* is constrained to move along a circle with centre *C* (circular motion).
- *P* appears to move in a straight line perpendicular to *OC*.



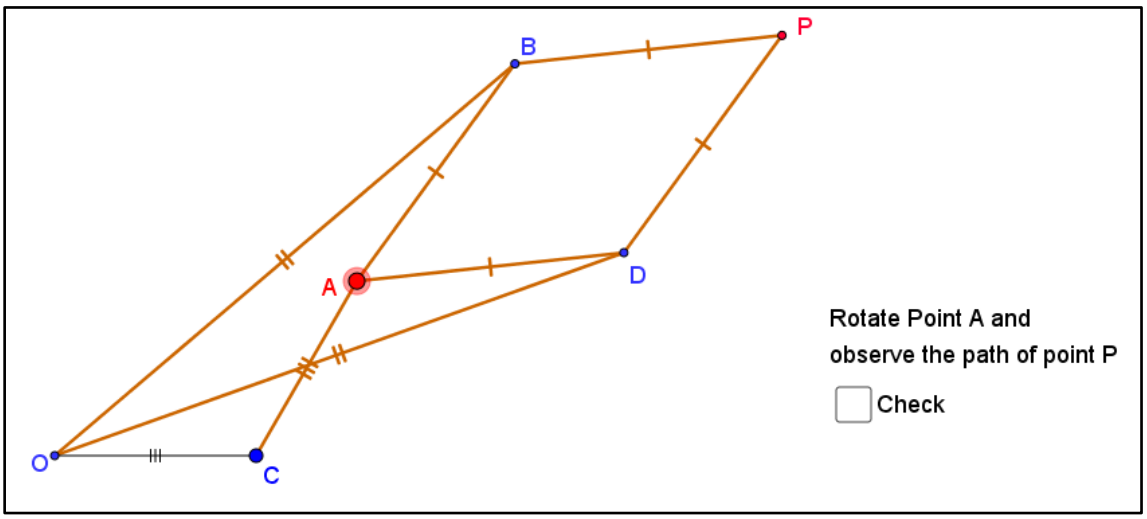


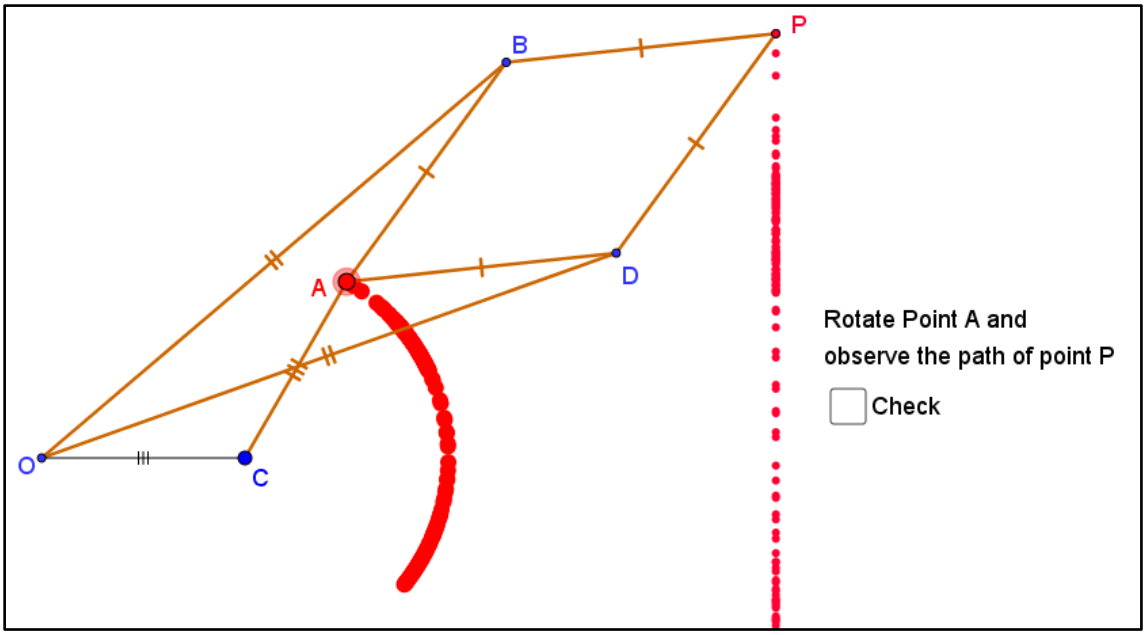
Using a computer simulation

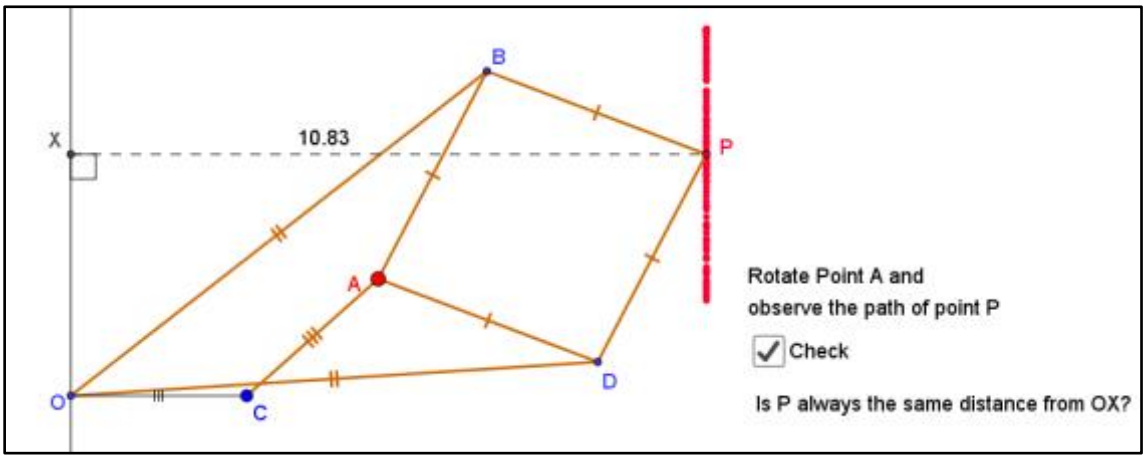
A GeoGebra simulation of Peaucellier's linkage is provided.

As point *A* is rotated (constrained along a circular path), the trace of point *P* shows that *P* appears to move in a linear path.

Using the check box shows the distance of *P* from the line *OX*. It appears that *P* moves along an exact straight line segment, unlike Chebycheff's linkage where only an approximation to a straight line was produced.

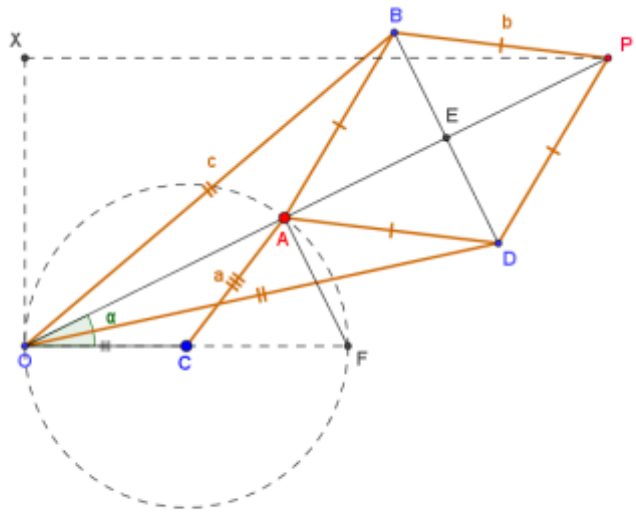






Proving that Peaucellier's linkage produces exact straight line motion

The proof traditionally includes inversion of a circle. However, an alternate proof is provided in [Teacher Sheet 1 - Peaucellier's Linkage Proof](#).



1. Prove that O, A and P are collinear.

2. Using right angled triangles OEB and BEP, (why are they right-angled?) show that OA.OP is constant.

3. Use △OAF and △PXO to prove that PX is constant.

OX is perpendicular to OC.

XP is perpendicular to OX

F is a point on the circle with centre C such that OF is a diameter of the circle

Conclusion

Peaucellier's linkage produces exact straight line motion from circular motion

Further Investigation

Can Peaucellier's linkage be used in reverse, to convert linear motion to circular motion?

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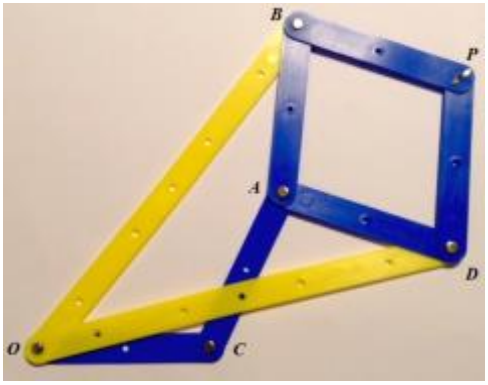
Peaucellier's Linkage

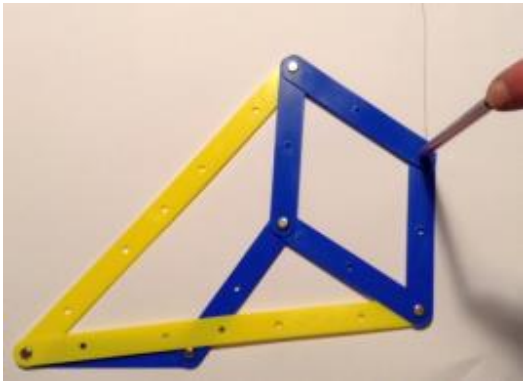
Name: _____

As a result of the industrial revolution in the 19th century, a number of mechanical linkages for converting circular motion into linear motion were designed, often by mathematicians. Some of these linkages produced *approximately* linear motion over part of the motion, whereas others produced exact linear motion. Peaucellier was a French engineer who invented the linkage in 1864.

Making a physical model

- Construct a model of Peaucellier's linkage from strips such as Geo Strips.
- See how it moves.
- Remove the paper fastener at point P and place a pencil through the holes. Observe the path of P as the linkage is rotated.



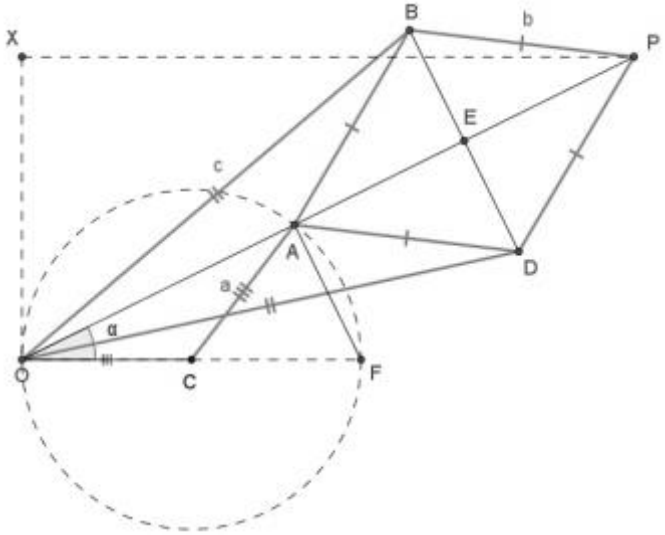


Using a computer simulation

- Open the GeoGebra file *Peaucellier*.
- Observe the path of *P* as *A* moves on a circular path.
- Check the distance *PX*. What is your conjecture?

Proving your conjecture

1. Prove that *O*, *A* and *P* are collinear.
2. Use right-angled triangles Δ*OEB* and Δ*BEP* (and their constant hypotenuses *c* and *b*) to show that $OA \times OP$ is constant.
3. Use Δ*OAF* and Δ*PXO* to prove that *PX* is constant and hence that *P* moves in an exact straight line.



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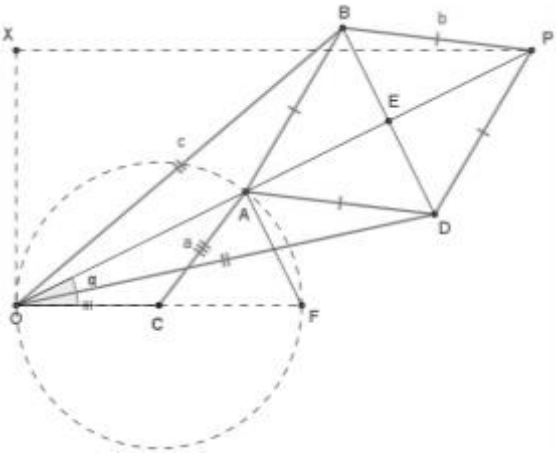
Teacher Sheet 1 – Peaucellier's Linkage Proof

Proving that *O*, *A* and *P* are collinear



Firstly it must be proved that *O*, *A* and *P* are collinear.

$\Delta OAB \equiv \Delta OAD$ (S.S.S.)

So $\angle OAB = \angle OAD$ (matching angles in congruent triangles)

$\angle BAP = \angle DAP$ (diagonal *AP* bisects rhombus angle $\angle BAD$)

So $\angle OAB + \angle BAP = \angle OAD + \angle DAP$

So $\angle OAB + \angle BAP = 180°$ and $\angle OAD + \angle DAP = 180°$

Hence O, A and P are collinear.

Proving that the motion of P is linear

In the diagram above, *OX* is perpendicular to *OC*, *XP* is perpendicular to *OX* and *F* is a point on the circle such that *OF* is a diameter. We need to show that *XP* has constant length.

In ΔBEP

$\angle BEP = 90°$ (diagonals of rhombus bisect each other at right angles)

$BE^2 = b^2 - EP^2$ (i)

In ΔOEB

$\angle OEB = 90°$ (diagonals of rhombus bisect each other at right angles)

$BE^2 = c^2 - OE^2$ (ii)

Combining (i) and (ii)

$b^2 - EP^2 = c^2 - OE^2$

So $OE^2 - EP^2 = c^2 - b^2$

$(OE - EP)(OE + EP) = c^2 - b^2$

$OA \times OP = c^2 - b^2$ (iii)

Let $\angle COP = \angle FOA = \alpha$

$OX \perp XP$ (construction)

So $\angle OPX = \alpha$

ΔOAF is similar to ΔPXO

So $\dfrac{PX}{OP} = \dfrac{OA}{FO}$

$OA \times OP = PX \times FO$

So $c^2 - b^2 = PX \times 2a$ (from iii)

So $PX = \dfrac{c^2 - b^2}{2a}$

Hence PX is constant and P moves along a straight line path perpendicular to OC.

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