Year 10: Proof

Lesson 5: Sylvester's Pantograph

Australian Curriculum: Mathematics (Year 10)

ACMMG243: Formulate proofs involving congruent triangles and angle properties.

- Applying an understanding of relationships to deduce properties of geometric figures (for example the base angles of an isosceles triangle are equal).

ACMMG244: Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes.

- Performing a sequence of steps to determine an unknown angle giving a justification in moving from one step to the next.
- Communicating a proof using a sequence of logically connected statements.

Lesson abstract

Students construct a physical model of Sylvester's pantograph; a drawing instrument designed for copying drawings. They explore how the copied image compares with the original drawing and use a computer simulation of the pantograph to explore how the geometric design of the pantograph allows the pantograph to work. Students then develop a geometric proof based on a sequence of deductive reasoning in which they use their knowledge of rhombus properties and congruent triangles.

Mathematical purpose (for students)

We can use logically connected geometric statements to explain why the object works the way it does.

Mathematical purpose (for teachers)

The lesson provides a novel but challenging way to engage students in geometric reasoning, leading to construction of a geometric proof. The geometry of the pantograph is based on properties of a rhombus and congruent triangles. An important feature of the proof is to introduce appropriate construction lines.

Lesson Length 100 minutes approximately

Vocabulary Encountered

- pantograph
- isometric
- construction line

Lesson Materials

- Geostrips or similar (6 identical strips and 2 short strips per pair)
- Paper fasteners (8 per pair of students)
- Sheet of A3 paper per pair of students
- Student Sheet 1 – Sylvester's Pantograph (1 per student)
- Slide show: *ST1_Yr10_5a_SylvesterImages.pptx*
- GeoGebra file *ST1_Yr10_5b_Sylvester.ggb* OR GeoGebra Tube https://ggbm.at/fwjYbYRd

We value your feedback after these lessons via https://www.surveymonkey.com/r/2JH6Z82

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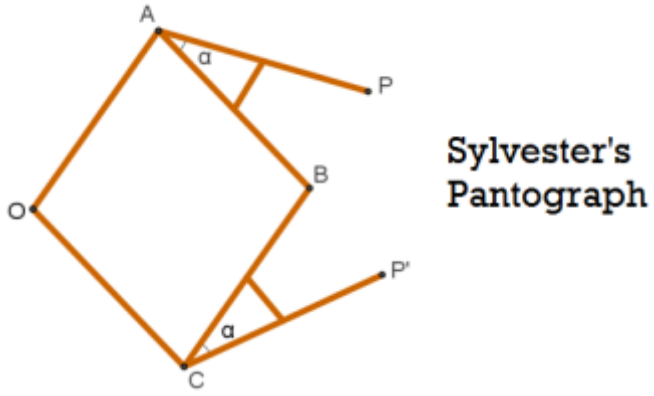
Introducing Sylvester's Pantograph

Teacher Notes

- James Sylvester was a 19th century English mathematician. Like many mathematicians at the time, he became interested in inventing machines based on mechanical linkages
- Unlike the enlarging pantograph, Sylvester's pantograph produces isometric (same size), but rotated, copies of drawings.
- The pantograph is constructed from six bars of equal length. Four of these bars form a rhombus. The point *O* is fixed. The short links between *AP* and *AB* and between *CB* and *CP'* ensure that ∠*BAP* and ∠*BCP'* are fixed and equal. When point *P* is moved around a shape, point *P'* traces out a congruent shape rotated through the fixed angle ∠*BAP* (which is also equal to ∠*BCP'*).



Getting started

- Show students either a real linkage or the initial part of the slide show *ST1_Yr10_5a_SylvesterImages.pptx*, encouraging them to observe both the size and position of the copy (the image of the original drawing).

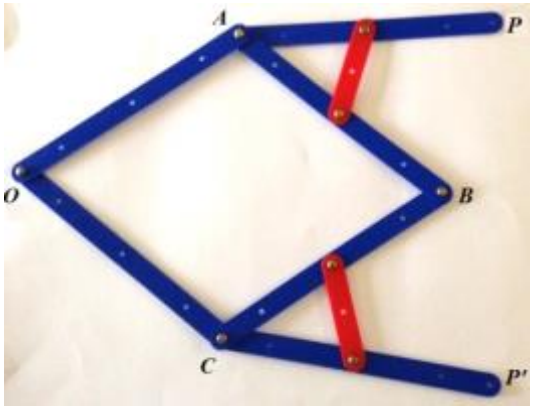
Expected Student Response

- The image drawn by the pantograph appears to be the same size as the drawing that is copied.
- The image is rotated.
- The image is rotated by approximately the fixed angle of the pantograph.
- Many line segments have the same length. The distance of *P'* from *O* appears to be always the same as the distance of *P* from *O*.

Models of the Pantograph

Making a physical model

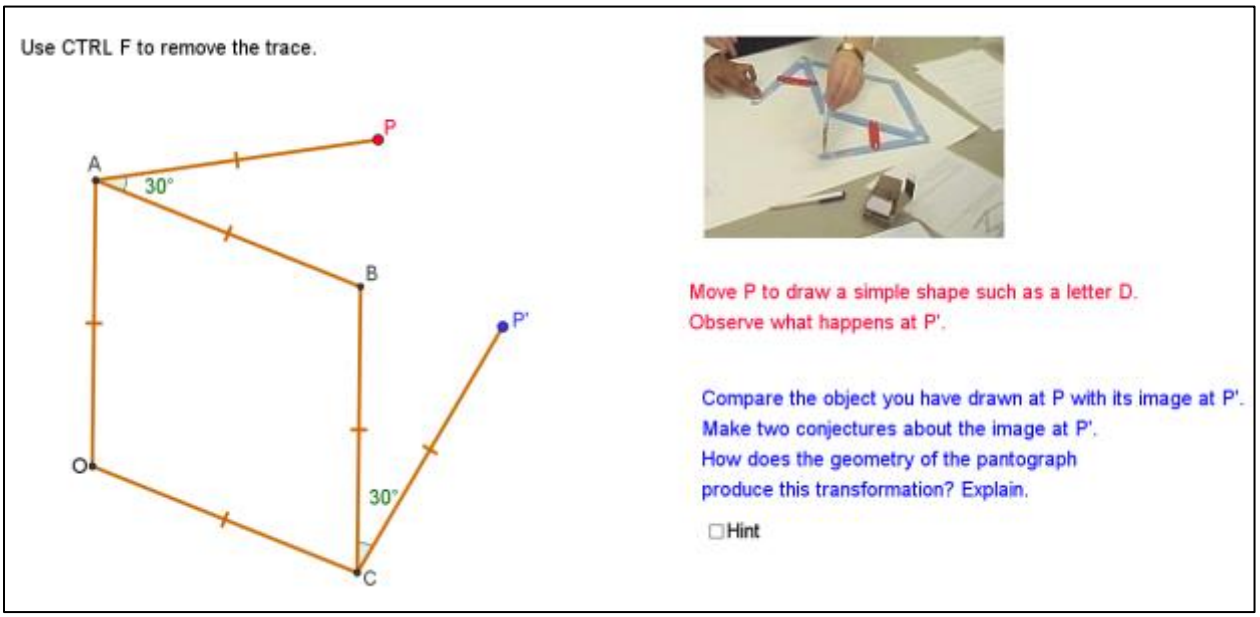
- Hand out Student Sheet 1 - Sylvester's Pantograph
- Students assemble the 6 identical strips as shown, with the 2 shorter strips being used to create two fixed but equal angles. Care needs to be taken to make sure the angles are equal.
- The paper fastener at pivot point *O* is inserted through the strips and through the piece of A3 paper so that the linkage is free to rotate about *O*.

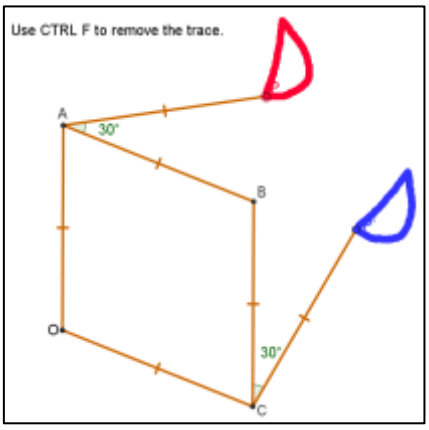


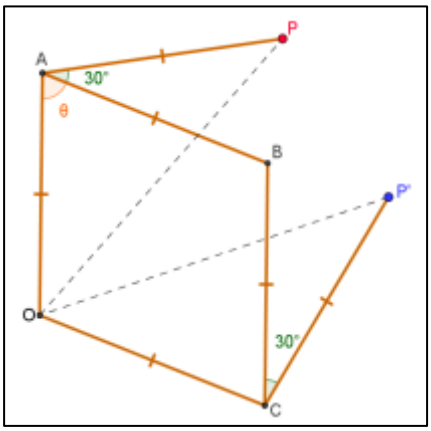
- Draw a large shape such as a letter D on the A3 paper. One student traces over the shape with point *P* while the other student lightly holds a pencil in the hole at *P'* and allows it to draw an image.
- The larger the shape, the more easily students will be able to compare the size and orientation of the image drawn at *P'*.
- Note: because there is some slack in the holes around the paper fasteners, students may have difficulty determining the precise relationship between the shape at *P* and its image at *P'*.

Using the dynamic geometry simulation

- The computer simulation of the pantograph allows students to make more precise comparisons between the shape they draw at *P* and the copy at *P'*.







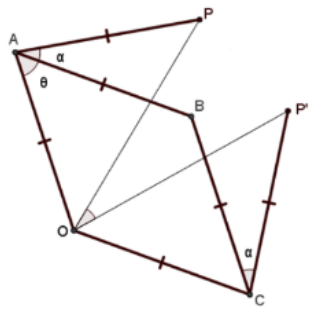
Developing an explanation for why the pantograph works the way it does

- Students may or may not be aware already that it is sometimes necessary to add construction lines to a geometric diagram when constructing a proof. Selecting the Hint checkbox in the GeoGebra simulation shows segments *OP* and *OP'* as well as labelling $\angle OAB$ as θ (see above).
- Working in pairs, students should be encouraged to develop their reasons using the computer simulation. They may notice properties that stay constant and properties that vary as they move point *P*, for example, although the rhombus changes shape, $\angle POP'$ stays the same, distances *OP* and *OP'* stay equal.
- Encourage students to turn their informal conjectures into geometric statements to be proved. For example,
    - The image at *P'* is the same size as the object drawn at *P*: $OP = OP'$
    - The image is rotated by the fixed angle of the pantograph: $\angle POP' = \angle PAB = \angle P'CB$

Constructing proofs

- Drawing a careful diagram and labelling with the same letters for angles that are known to be equal helps students to develop a logical sequence of reasoning.

- Some students may be able to develop their own proofs of the two conjectures. Others, though, may benefit from a revisiting of the computer simulation as a whole class activity.



- Students may suggest that isosceles triangles *OAP* and *OCP'* are congruent. Why?
  - Opposite angles of rhombus are equal so $\angle ACB = \alpha$.
  - So $\angle OAP = \angle OCP'$
  - So triangles *OAP* and *OCP'* are congruent (S.A.S.), and hence *OP* = *OP'*
- Proofs of the two key conjectures are given on Teacher Sheet 1 – Two Proofs.
- Advice on structuring the proving process for students is given on Teacher Sheet 2 – Structuring the Proving Process

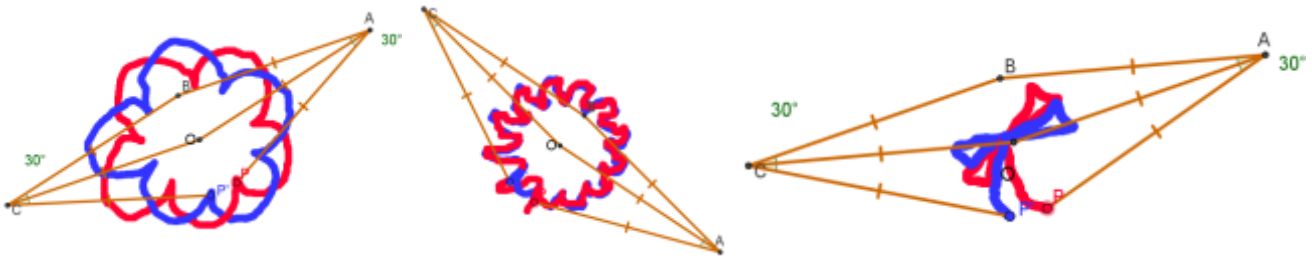
Conclusion

- Starting with known facts and knowledge of geometric properties, a geometric argument can be constructed as a sequence of logically connected statements. Geometry can be used to explain both the size and orientation of the image produced by the pantograph.

Further Investigations

Making Patterns

Use the pantograph (physical model or computer simulation) to make some nice patterns. Explain their geometric properties and how your pattern depends on the fixed angle of the pantograph.



Would a parallelogram version of Sylvester's pantograph work?

In fact, Sylvester's pantograph works in the same way for any parallelogram, not just a rhombus. Students construct a physical model and test it. A proof is provided in Teacher sheet 3: Parallelogram Pantograph

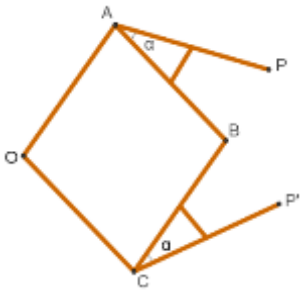
Constructing a parallelogram version of the pantograph in dynamic geometry

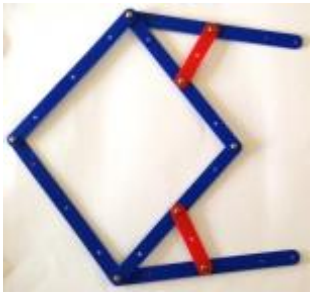
The construction method is very similar to that for the rhombus version. An important aspect of simulations of all mechanical linkages is that the point that is required to be dragged must be constructed as an independent point. The construction must then be completed based on this independent point. Advice is given on Teacher Sheet 4 – Parallelogram Construction.

Sylvester's Pantograph

Name: _____

Drawing instruments known as pantographs were widely used to copy or enlarge drawings before the development of photocopiers and computers. James Sylvester was a 19th century English mathematician. Sylvester's pantograph is constructed from six equal links: *OA*, *OC*, *AB*, *BC*, *AP* and *CP'*, where point *O* is fixed. The short links between *AP* and *AB* and between *CB* and *CP'* ensure that ∠*BAP* and ∠*BCP'* are fixed and equal.





Making a physical model

- Construct the pantograph from 6 identical strips, 2 shorter strips and 8 paper fasteners.
- Make a small hole in a piece of A3 paper (or thin card) and use a paper fastener to fix point *O* at the left side of the paper so that the pantograph is free to rotate about *O*, as shown above.
- Draw a shape such as a large letter D on the paper near point P.
- Place a pencil in the hole at *P'* and hold lightly while you trace over the shape with point *P*. This will need one person to move point *P* over the drawn shape and the other person to hold the pencil so that it can follow the path traced by point *P*.
- Compare the image drawn by the pencil at *P'* with the shape traced over by point *P*. Make two conjectures about the image at *P'* compared with the shape at *P*, one conjecture relating to the size of the image and the other relating to the orientation of the image.

Using a computer simulation

- Open the GeoGebra file *Sylvester* and drag point *P* to create a shape, e.g., a letter D.
- Check or refine your conjectures.

Prove your conjectures

- Draw a diagram of the pantograph similar to the above diagram and use the same letters to label angles that you know are equal. It may be useful to draw segments *OP* and *OP'*
- Using a sequence of logically connected geometric statements, construct proofs for each of your conjectures. You could set them out with these headings:

1. Size of image

Given: $OA=CB=CP=OC=AB=AP'$, $\angle BCP = \angle BAP' = \alpha$,
Prove:
Proof:

2. Rotation of image

Given: $OA=CB=CP=OC=AB=AP'$, $\angle BCP = \angle BAP' = \alpha$,
Prove:
Proof:

A Design Question

- Prove that the pantograph would still work in the same way if *OABC* were a parallelogram rather than the special case of a rhombus.
- Construct a parallelogram version of Sylvester's Pantograph and try it out.

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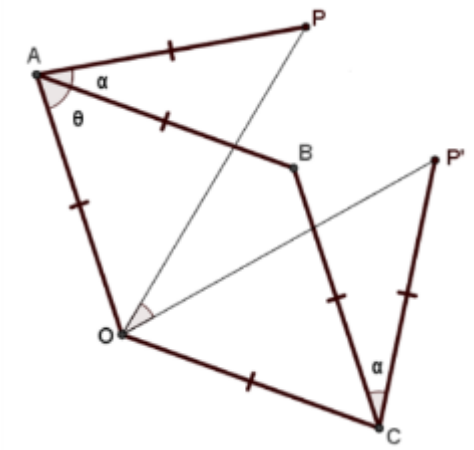
Teacher Sheet 1 - Two Proofs



Two conjectures

1. $OP = OP'$
2. $\angle POP' = \angle PAB$ $(=\angle P'CB)$

Proof that $OP = OP'$

Let the fixed angles $\angle BAP$ and $\angle BCP'$ be α.

In ΔOAP and $\Delta OCP'$,

$\angle OAB = \angle OCB$ (rhombus $OABC$)

So $\angle OAP = \angle OCP'$ (equal rhombus angles plus same fixed angle α)

$OA = OC, AP = CP'$ (given)

So $\Delta OAP \equiv \Delta OCP'$ (SAS)

So $OP = OP'$

Proof that $\angle POP'$ remains constant and equal to $\angle PAB$

In isosceles triangles, ΔOAP and $\Delta OCP'$,

$$\angle AOP = \angle COP' = \frac{1}{2}(180° - \theta - \alpha)$$

So $\angle AOP + \angle COP' = 180° - (\theta + \alpha)$

$$\begin{aligned}\angle AOC &= \angle AOP + \angle COP' + \angle POP' \\ &= 180° - (\theta + \alpha) + \angle POP'\end{aligned}$$

But in rhombus $OABC$,

$\angle AOC = 180° - \theta$ (adjacent angles of rhombus are supplementary)

So $180° - (\theta + \alpha) + \angle POP' = 180° - \theta$

So $\angle POP' = \alpha$

Hence, since OP and OP' remain equal and $\angle POP'$ remains constant, the movement of P' is a copy of the movement of P, but rotated through an angle α.

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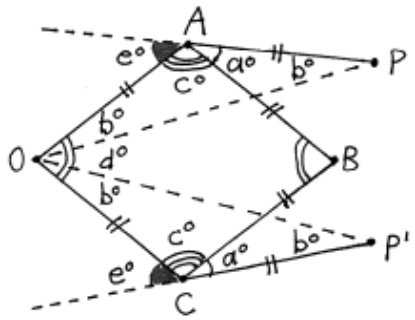
Teacher Sheet 2 - Structuring the Proving Process

In the lesson plan, the same unstructured approach has been taken to this task as for the *Consul* investigation. By now, students should be developing an understanding of conjecturing and proving and how a proof is constructed from a logical sequence of steps of reasoning which are justified on the basis of previously accepted understanding.

Drawing a careful diagram and labelling with the same letters for angles that are known to be equal helps students to develop a logical sequence of reasoning (see students' drawing below).



Whilst some students may be able to develop a proof without assistance, others may be assisted with a series of questions.

Size of image

- What geometric shape is quadrilateral *OABC*? Explain.
- In Δ*OAP* and Δ*OCP'*, what can you say about ∠*OAP* and ∠*OCP'*? Explain.
- What can you now say about Δ*OAP* and Δ*OCP'*? Explain.
- What does this tell you about *OP* = *OP'*?

Rotation of image

- From the given information what do you know about ∠*PAB* and ∠*P'CB*?
- From the given information what do you know about ∠*OAB* and ∠*OCB*? Explain.
- What can you say about the four angles ∠*AOP*, ∠*APO*, ∠*COP'*, ∠*CP'O*? Explain.
- Show that ∠*POP'* = ∠*PAB* (or ∠*P'CB*).

Samples of students' conjecturing and deductive reasoning

Included below is a sample of the reasoning and proofs of two advanced Year 8 students.

Students 1 and 2

The pair of students assembled their pantograph from strips using the diagram on their worksheets and worked together on a piece of A3 paper. They tested the movement of the pantograph to make sure their initial drawing at point *P* would allow the image to fit on the paper. They were confident that the image was the same size as their original drawing, and Student 2 conjectured that the image had been rotated through 45 degrees, basing her conjecture on a visual impression. The laugh that accompanied her statement indicated that she was not confident of the accuracy of this conjecture:

Student 1: It's the same but it's not in the same direction. It's been turned.

Student 2: [spreading her thumb and fingers to compare the size] It's the same size. It's turned 45 degrees.

Student 1: How do you know it's 45 degrees?

Student 2: Just guessing, 'cause it was about half 90.

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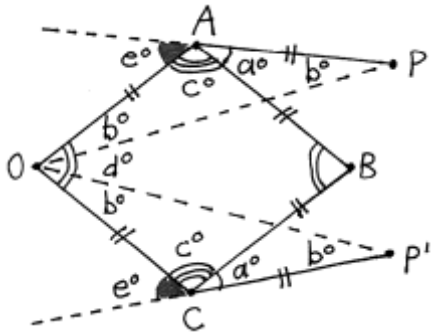
Conjecture 1:
Make a conjecture about the size of the image compared with the size of the original shape.
The image & shape are congruent

Conjecture 2:
Make a conjecture about the rotation of the image compared with the position of the original shape.
The image rotates at the degree of ∠BAP or ∠BCP



Student 1: Proof, OK.

Student 2: Let's do the sides first. *OA* equals *AP* equals *OC* equals *CP'* … then angle *OCP* equals … *OPB* because they both have 30 degrees … they share 30 degrees … we shouldn't do that yet. Angle *OA* … angle *OAB* equals …

Student 1: Angle *OCB*

Student 2: And angle *BCP'* equals *BAP* because given … *OAB* plus *BCP'* …

Student 1: Those two added together, that whole angle … that means …

Student 2: Once we've proved that angle, then the whole thing's easy 'cause side angle side … see, if you have two sides and how big it's going to be in between … when you join them up the triangles will be the same…

Student 1: Oh, yep. So … angle *P'CO* will be equal to ...

Student 2: Therefore … *P'CO* equals *PAO* because … say side angle side so it makes congruent triangles. So *OP* equals *OP'*.

Student 1: Right, now prove that *POP'* equals angle *P'CB* and *PAB*. In triangle *POA* …

Student 2: We have to add them first … angle *POA* is made up of angle *C*

Student 2: And the reason is … angles in triangle … and now if *AO* … oops … if 2*b* plus *c* plus *a* equals 180 and 2*b* plus *c* plus *d* equals 180 because they're supplementary in a rhombus … then … OK … now how do we write this out?

Student 1: I'm all confused.

Student 2: *b* plus *b* plus *d* plus *c* equals180 because they're two angles in a rhombus.

Student 1: Oh, yep …

Student 2: And *b* plus *b* plus *a* plus *c* equals 180 'cause triangle …

Student 1: … Yep …

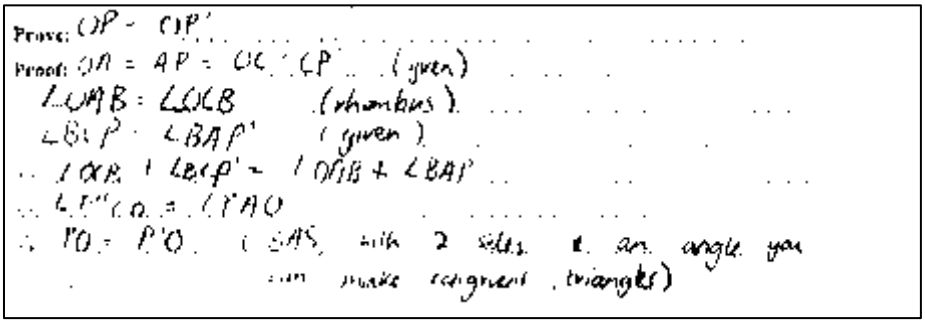
Student 2: So *a* equals *d*.

Student 1: Yep.

The students' written proofs

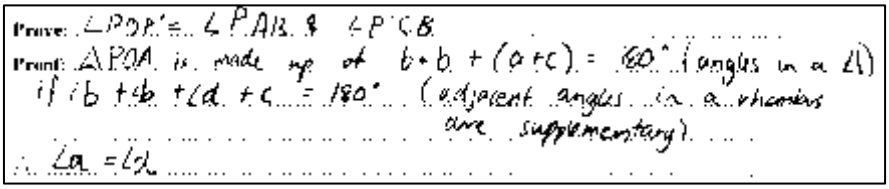
(a) Size

Given: $OA = AB = BC = CO = AP = CP'$; $\angle BAP = \angle BCP' = \alpha$


Prove: OP = OP'
Proof: OA = AP = OC = CP' (given)
∠OAB = ∠OCB (rhombus)
∠BCP' = ∠BAP' (given)
∴ ∠OAB + ∠BAP' = ∠OCB + ∠BCP'
∴ ∠P'CO = ∠PAO
∴ PO = P'O (SAS, with 2 sides & an angle you can make congruent triangles)

(b) Rotation

Given: $OA = AB = BC = CO = AP = CP'$; $\angle BAP = \angle BCP' = \alpha$


Prove: ∠POP' = ∠PAB & ∠P'CB
Proof: △POA is made up of b + b + (a + c) = 180° (angles in a △)
if ∠b + ∠b + ∠d + c = 180° (adjacent angles in a rhombus are supplementary)
∴ ∠a = ∠d

Although there are small mistakes and omissions, for example, the statement $\angle BCP = \angle BAP'$ in line 3 of the first proof should be $\angle BAP = \angle BCP'$, the proofs demonstrate a logical ordering of statements. Apart from temporary confusion when Student 2 stated several steps of reasoning at once, Student 1 seemed to follow the reasoning. As well as understanding the geometry of the linkage, Student 1 and Student 2 also seemed to appreciate how the pantograph was operating:

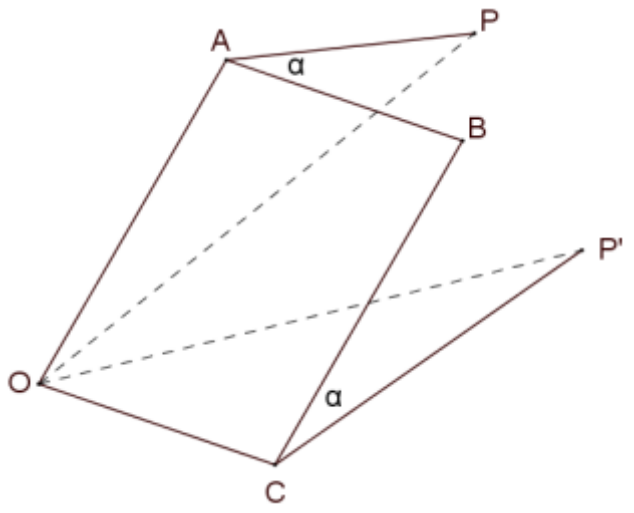
Teacher: Can you explain now why the angle of rotation that you noticed there is equal to this angle [*POP*'].

Student 1: Oh, because that's [pointing to *OP* and *OP*'] opened up by that amount and it …

Student 2: And as it [*P*] moves up it [*OP*'] turns around by that angle.

Teacher Sheet 3 - Parallelogram Pantograph

Proof that a Sylvester-like pantograph can be based on a parallelogram.



1. Proof that $OP = OP'$

   In ΔOAP and $\Delta OCP'$,

   $OA = CP', AP = OC$ (given)

   $\angle OAB = \angle OCB$ (parallelogram $OABC$)

   $\angle OAP = \angle OCP'$ (equal angles plus same fixed angle, α)

   $\Delta OAP \equiv \Delta OCP'$ (S.A.S.)

   So $OP = OP'$ (matching sides of congruent triangles)

2. Proof that $\angle POP'$ remains constant and is equal to the fixed angle, α

   In parallelogram $OABC$

   $\angle AOC + \angle OAB = 180°$ (adjacent angles of parallelogram)

   So $\angle AOP + \angle POP' + \angle COP' + \angle OAB = 180°$

   But in ΔAOP,

   $\angle AOP + \angle OAB + \alpha + \angle APO = 180°$

   So $\angle AOP + \angle POP' + \angle COP' + \angle OAB = \angle AOP + \angle OAB + \alpha + \angle APO$

   But $\angle COP' = \angle APO$ ($\Delta OAP \equiv \Delta OCP'$)

   So $\angle POP' = \alpha$

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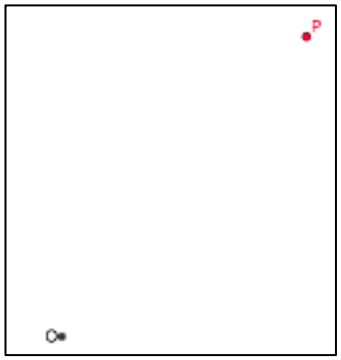
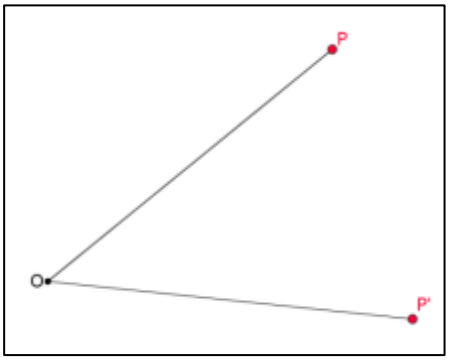
Teacher Sheet 4 - Parallelogram Construction

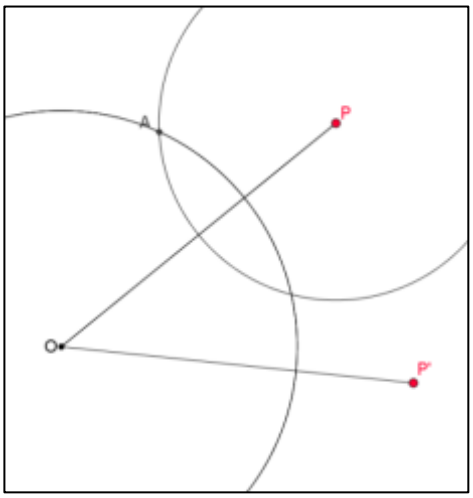
Constructing a parallelogram version of Sylvester's Pantograph in GeoGebra

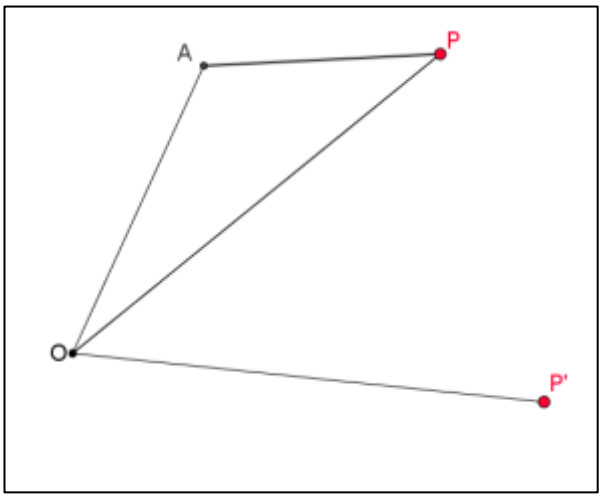
- Construct a point *O* and fix the point. Construct another point *P* but do not fix this point.
- Rotate *OP* about *O* through a given angle in a clockwise direction. Here the angle of rotation was 45°.
- Hide the label for point *O*'





- Using the *Circle with Centre and Radius* tool, construct a circle with centre *O*. Here, the circle had radius 8 cm.
- Construct a second circle with centre *P* having a different radius. In this case the radius was 6 cm.
- The radii of these two circles form the sides of the parallelogram. Construct an intersection point, *A*, where the two circles intersect. If your circles do not intersect, change the radii so they do intersect.
- Construct segments *OA* and *AP*.





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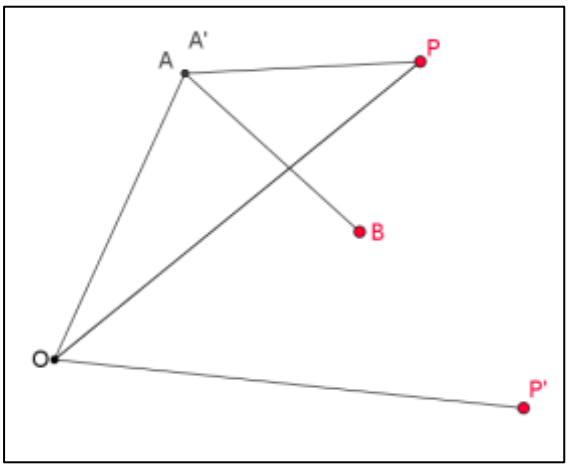
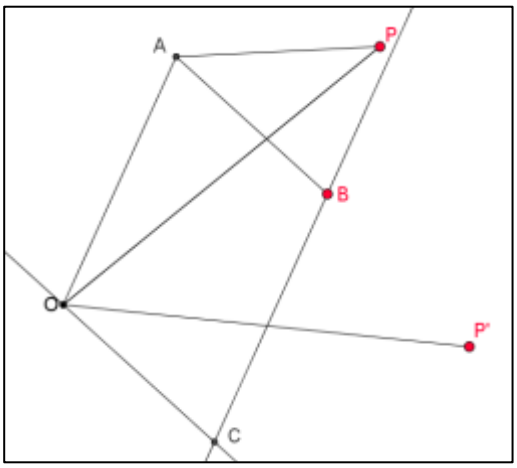
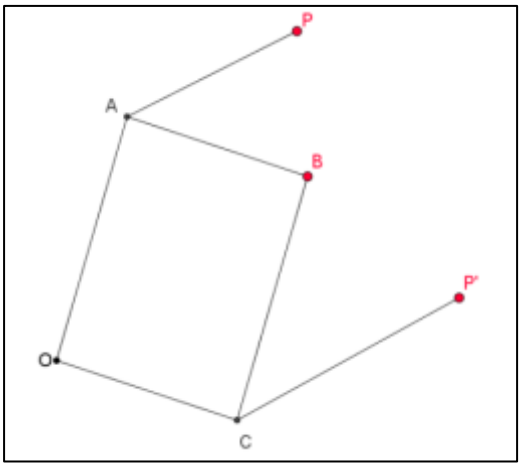
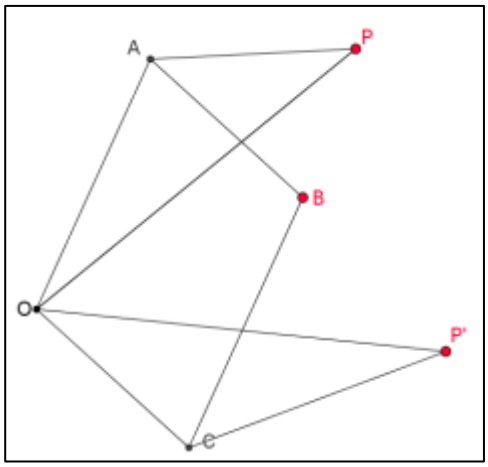
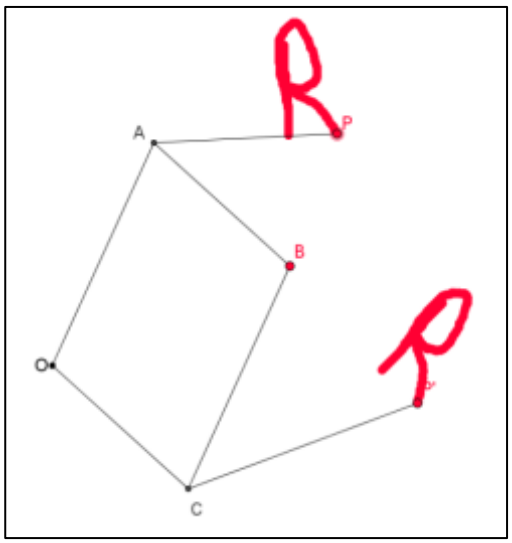
- Rotate *AP* about point *A* in a clockwise direction through the same angle as used before (in this case, 45°). Hide the label of rotated point *A*'. Rename the rotated point P_1' as point *B*.
- Construct a line through *O* parallel to *AB*.
- Construct a line through *B* parallel to *AO*.
- Construct an intersection point where the parallel lines intersect. Rename this point *C*.





- Hide the parallel lines and construct segments *BC*, *OC* and *CP*'.
- Hide *OP* and *OP*'.
- Switch on Trace for points *P* and *P*' and test the pantograph.