Year 10: Proof
Lesson 1: Chebycheff's Linkage

Australian Curriculum: Mathematics (Year 10)

ACMMG243: Formulate proofs involving congruent triangles and angle properties.

- Applying an understanding of relationships to deduce properties of geometric figures (for example the base angles of an isosceles triangle are equal).

ACMMG244: Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes.

- Performing a sequence of steps to determine an unknown angle giving a justification in moving from one step to the next.
- Communicating a proof using a sequence of logically connected statements.

Lesson abstract

Students investigate Chebycheff's linkage and determine whether it produces approximate or exact linear motion from circular motion. They make a physical model, observe the motion of points and try to explain them, and then observe a computer simulation. Visually, the motion appears as a very convincing straight line, so students are misled until measurements expose the small variation from linear motion. This establishes awareness of the need for proof in other geometry contexts. Students can also use Pythagoras' theorem to establish some lengths.

Mathematical purpose (for students)

Seeing is not always believing. Proof must be based on a logical argument, rather than on visual evidence.

Mathematical purpose (for teachers)

The lesson establishes awareness of the need for geometric proof by creating a situation where visual evidence is misleading. As a result of the industrial revolution in the 19th century mathematicians became involved in the development of mechanical linkages for converting between circular motion and linear motion. Some of these linkages produced exact linear motion whereas others (including Chebycheff's) produced *approximately* linear motion over part of the motion. In addition to establishing the need for proof, there are opportunities for relatively simple reasoning about the paths of points on linkages and for calculating distances using Pythagoras' theorem in multi-step arguments.

Lesson Length 30 minutes approximately

Vocabulary Encountered

- Circular motion
- Linear motion

Lesson Materials

- Per pair: Geostrips (2 long yellow, 1 long and 1 short blue); 4 paper fasteners; piece of card slightly larger than A4; piece of A4 paper
- Student Sheet 1 – Chebycheff's Linkage (1 per student)
- Slide show: *ST1_Yr10_1a_ChebycheffImages.pptx*
- GeoGebra file: *ST1_Yr10_1b_Chebycheff.ggb* OR GeoGebra Tube https://ggbm.at/KAHjfecN (teacher only)
- Teacher to make a model to bring to class

We value your feedback after these lessons via https://www.surveymonkey.com/r/2JH6Z82

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Linear and Circular Motion

Teacher Notes

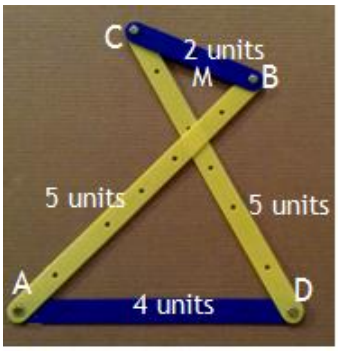
When introducing students to the concept of geometric proof, it is useful to establish a need for proof by creating a situation where visual evidence misleads students. This lesson uses Chebycheff's linkage to highlight this point.

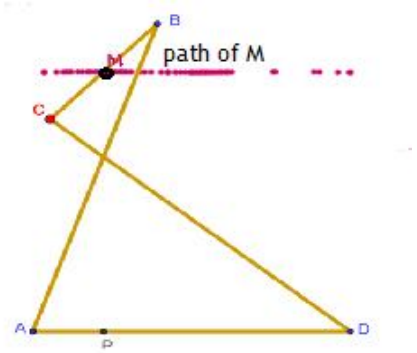
The industrial revolution began in the mid 1700's in Britain. The steam engine became the main source of power, replacing water wheels and horses, and machines were invented to do repetitive tasks previously done by people. Many of these engines and machines needed to convert circular motion to linear motion and vice versa. For example, James Watt discovered how to convert the linear motion obtained by the expansion and condensation of steam in a cylinder into rotation of a drive shaft to make a better steam engine. The pistons and crank system in a modern car serve a similar purpose, rotating the drive shaft to take the power to the wheels.

The flourishing of mechanical invention that occurred as a result of the industrial revolution led to the development of a number of linkages for converting circular motion into linear motion. Some of these linkages, for example those of Watt and Chebycheff, produced *approximately* linear motion over part of the motion, whereas others, for example Peaucellier's linkage, produced exact linear motion. Simple three-bar or four-bar linkages which produced approximate linear motion had the advantage of less moving parts, and hence less wear, whereas the linkages which produced true linear motion were less practical, and have survived only as mathematical curiosities.

The Russian mathematician Chebycheff (1821–1894) designed a linkage consisting of four bars, two of which cross over each other as shown below. As the linkage is rocked backwards and forwards, the midpoint of the top bar moves in a very close approximation to a straight line.

There is a lot of information about linkages that convert rotational motion to linear motion (and vice versa) on the internet. (Note that Chebycheff is sometimes transliterated as Tchebyshev or Tchebycheff.)





Getting started

- Explain a little of the historical background, and the need to convert between circular and linear motion.
- Have students share their knowledge about how this is done in cars, motor bikes or other machines.
- Show a model of Chebycheff's linkage that you have prepared in advance, and demonstrate its movement.
- Explain that the aim of the lesson is for them to investigate how (or whether) the linkage converts between linear and circular motion. Do not give away the fact that the motion is only approximately linear.
- Useful images for the lesson are supplied in the slide show *ST1_Yr10_1a_ChebycheffImages.pptx*.

Models of Chebycheff's Linkage

Students make a physical model

- This linkage is simple to construct from plastic Geo Strips or homemade card strips. Apparent linear motion is readily seen using a Geo Strip model using two long yellow strips, a short blue strip and a long blue strip. Note that a model constructed with Geo Strips will not have the exact ratio of lengths as in Chebycheff's linkage but it still generates a convincing 'straight' line.

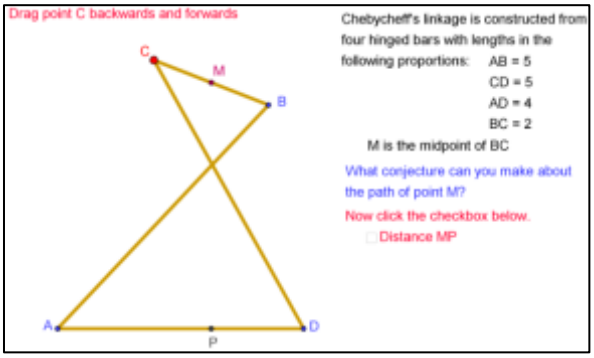
- The detailed instructions for making the linkage are on Student Sheet 1 – Chebycheff's Linkage.
- Students trace the motion of at least 3 points as the linkage is rocked back and forward, describe the motion of each and try to explain geometrically.
- Discuss students' observations and ideas.

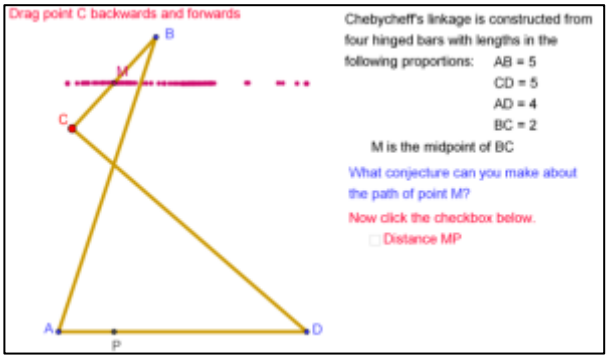
Expected student responses

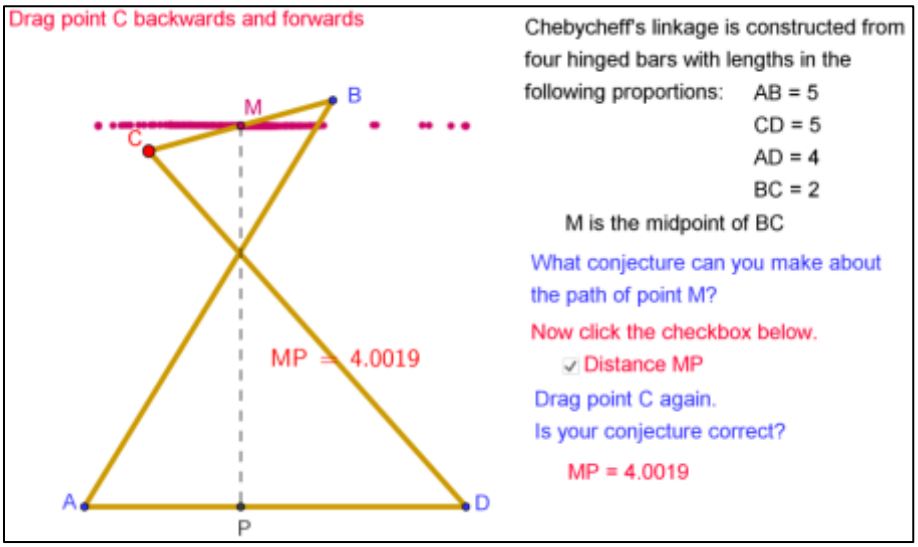
- The points on the strip *AD* do not move.
- The points on the strip *AB* move in circular paths, with *A* as the centre. Reason: each point keeps a constant distance from the fixed point *A*.
- The points on the strip *CD* move in circular paths, with *D* as the centre. Reason: similar.
- The points on the strip *BC* generally move along curved paths, except for the midpoint which moves in a straight line parallel to *AD*. Reason: unclear.
- Some students may note that the midpoint (M) is 4 units above line *AD*.

Demonstrate the computer simulation

- Pose the question of whether the midpoint really moves in a straight line, and why.
- Demonstrate the computer simulation, dragging point *C* to move the linkage. This will reinforce students' conjecture that *M* moves in a straight line.
- Discuss student ideas BEFORE clicking checkbox to measure *MP*.
- The computer simulation enables the path of *M* to be observed more precisely than with the physical model, and measured by clicking the checkbox. This demonstrates that the path of *M* is an extraordinarily close approximation to a straight line (over the relevant interval).
- Discuss how geometric claims must be proved by logical argument, not by appearance.

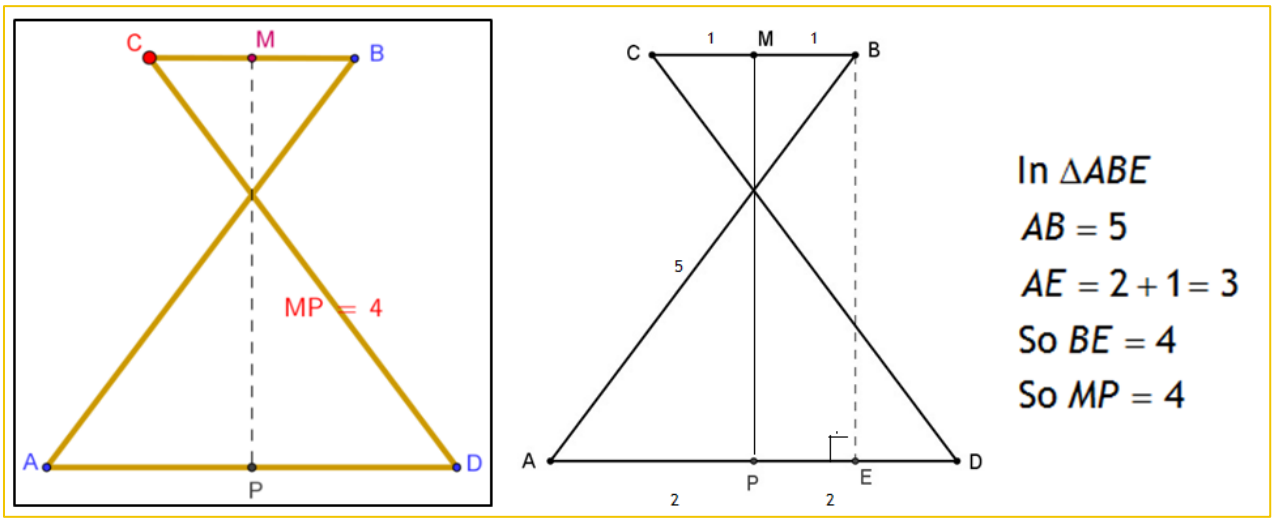


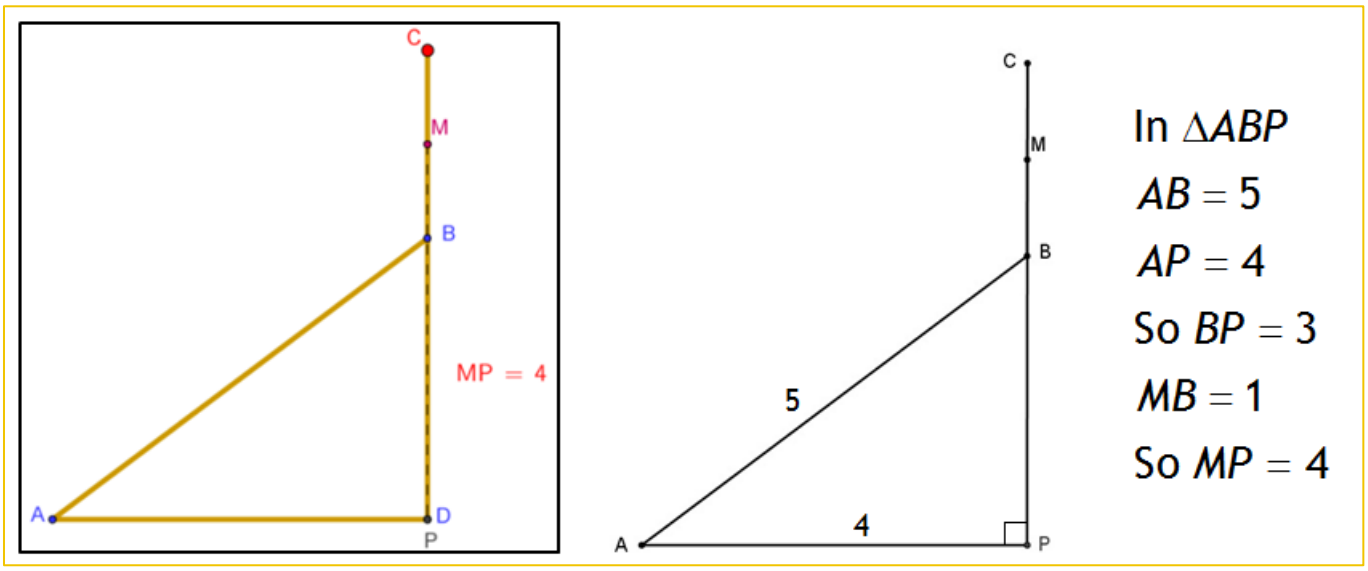




Explaining the special case positions

- In three positions the distance *MP* is exactly 4 units. In the other positions it is nearly 4 units.
- Use the simulation for class to locate these three positions (*P* at *A*, *P* at *B* and *P* at midpoint of *AB*).
- Set students the task of proving that *MP* is exactly 4 units in these three positions.





In the mirror image position when *P* is at *A*, *MP* is also exactly 4, and the argument is the same, but the simulation does not stretch that far to check this.)

Conclusion

Operating Chebycheff's linkage, where the more precise, although still empirical, feedback from the computer simulation conflicts with the visual evidence from the physical model, serves to sow a seed of doubt in the students' minds. In geometry: 'seeing is not always believing'.

Students could write an account of their experience with this linkage that explains what they now have proved and what they have learned about geometry.

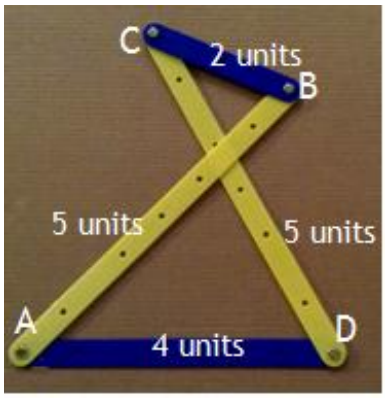
Chebycheff's Linkage

Name: _____

The industrial revolution began in the mid 1700's in Britain. The steam engine became the main source of power, replacing water wheels and horses, and machines were invented to do repetitive tasks previously done by people. Many of these engines and machines needed to convert circular motion to linear motion and vice versa. Many devices to do this were invented.

The Russian mathematician Chebycheff (1821–1894) designed a linkage for this purpose. It consists of four bars, two of which cross over each other as shown below.

Making a model



- You need 4 Geo Strips and 4 paper fasteners, a piece of A4 paper, and a piece of cardboard larger than A4.
- Choose the Geo Strips so that two are 5 units long (yellow in picture), one is 4 units long (blue) and one is 2 units long (blue).
- Connect the four strips as shown, with the two paper fasteners at points *A* and *D* (the ends of the long blue strip) passing through a piece of card as well as the blue and yellow strips.
- The two upper paper fasteners should be free to rock the linkage backwards and forwards.
- Place a piece of A4 paper under the linkage as shown.
- Place a pencil in one of the holes in each of the yellow strips and rock the linkage backwards and forwards, leaving a trace on the paper.
- Write down your observations for each hole and explain why you think the observations are true.
- Place a pencil in the hole in the centre of the short blue strip and the rock the linkage backwards and forwards.
- Write down your observations and explain why you think the observation is true.

Finding distances

- In some positions of the linkage, Pythagoras' theorem can be used to show that the distance of the midpoint *M* is exactly 4 units above *AD*. Find these positions and write down the reasoning.

Conclusion

- Write a summary of what you have learned about Chebycheff's linkage.

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