Year 10: Proof
Lesson 2: Scott Russell Linkage

Australian Curriculum: Mathematics (Year 10)

ACMMG243: Formulate proofs involving congruent triangles and angle properties.

- Applying an understanding of relationships to deduce properties of geometric figures (for example the base angles of an isosceles triangle are equal).

ACMMG244: Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes.

- Performing a sequence of steps to determine an unknown angle giving a justification in moving from one step to the next.
- Communicating a proof using a sequence of logically connected statements.

Lesson abstract

Students investigate the design and operation of a car jack based on the Scott Russell linkage. As the horizontal screw is turned, the car attachment point moves vertically (perpendicular to the screw). Students make a physical model and operate a computer simulation. Students then use the geometry of the linkage to find if the motion is really vertical. This involves reasoning about angles in three connected triangles.

Mathematical purpose (for students)

Is it a right angle or does it just look like a right angle? Angles in isosceles triangles can be used to find out.

Mathematical purpose (for teachers)

Students investigate the design and operation of a car jack based on the Scott Russell linkage which produces exact straight line motion perpendicular to another straight line motion. The design is based on two isosceles triangles forming a third triangle. Students use the geometry of the linkage to find if the apparent right angle determining the vertical motion is actually a right angle. The proof depends on the angle property of isosceles triangles and that the sum of the three angles of a triangle is 180°. The proof can be linked with the proof that the angle subtended by the diameter of a circle is a right angle.

Lesson Length 50-60 minutes approximately

Vocabulary Encountered

- perpendicular

Lesson Materials

- Car jack of Scott-Russell type if available.
- Per 2 students: 1 long strip and a second strip half the length, 2 paper fasteners, piece of A4 card, ruler.
- Slide show: *ST1_Yr10_2a_ScottRussellImages.pptx*
- GeoGebra file: ST1_Yr10_2b_ScottRussell.ggb OR GeoGebra Tube https://ggbm.at/HhTbn7qz
- Student Sheet 1 – Scott Russell Linkage (1 per student)

We value your feedback after these lessons via https://www.surveymonkey.com/r/2JH6Z82

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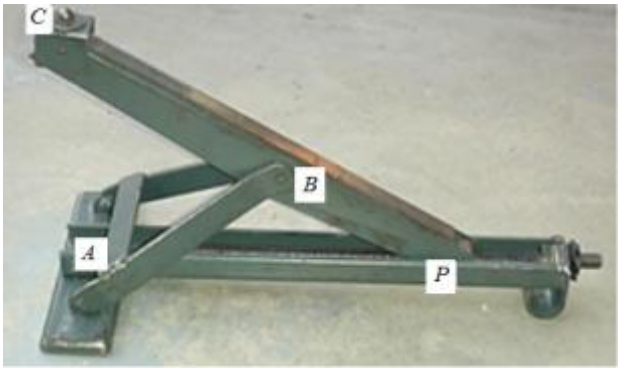
Introducing the Scott Russell Linkage

The Scott Russell linkage is named after the Scottish naval architect John Scott Russell. The linkage was originally patented in 1803 by English watchmaker William Freemantle.

The Scott Russell linkage has applications in the manufacturing and automotive industries (e.g. in car suspensions) and in robotics. See, for example, https://en.wikipedia.org/wiki/Scott_Russell_linkage. The car jack shown below is a simple example. The Scott Russell linkage was used in the design of a car jack supplied with certain cars, including the Ford Cortina and Lotus Elan, in the 1970s. This type of jack can sometimes be found in car wreckers' yards.

The geometry of the linkage is based on two isosceles triangles with the same side length for their equal sides. Referring to the illustration below, the linkage is constructed so that $AB = BC = BP$. As P moves closer to A, point C moves such that AC is perpendicular to AP.



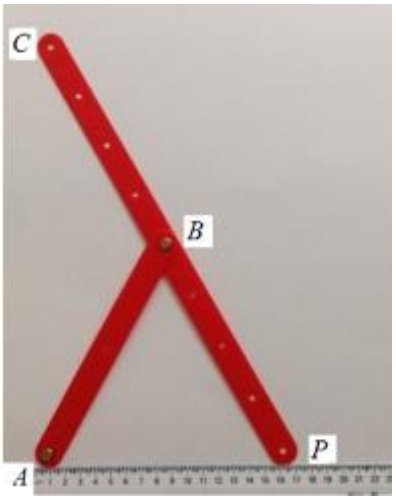
Getting started

- Show students the car jack if you have one, or images from the slide show *ST1_Yr10_2a_ScottRussellImages.pptx* and demonstrate how it works. As the horizontal screw at the base of the jack is turned so that point B moved closer to point A, the car attachment point P at the top of the jack moves vertically upwards.
- Discuss the important feature of the linkage in a car jack - to raise the car vertically. This important safety feature is the focus of the lesson.
- Show students the video of the the 'pick and place' robot to demonstrate other uses of this linkage. https://www.youtube.com/watch?v=PgQvw_9TFJU

Models of the Scott Russell Linkage

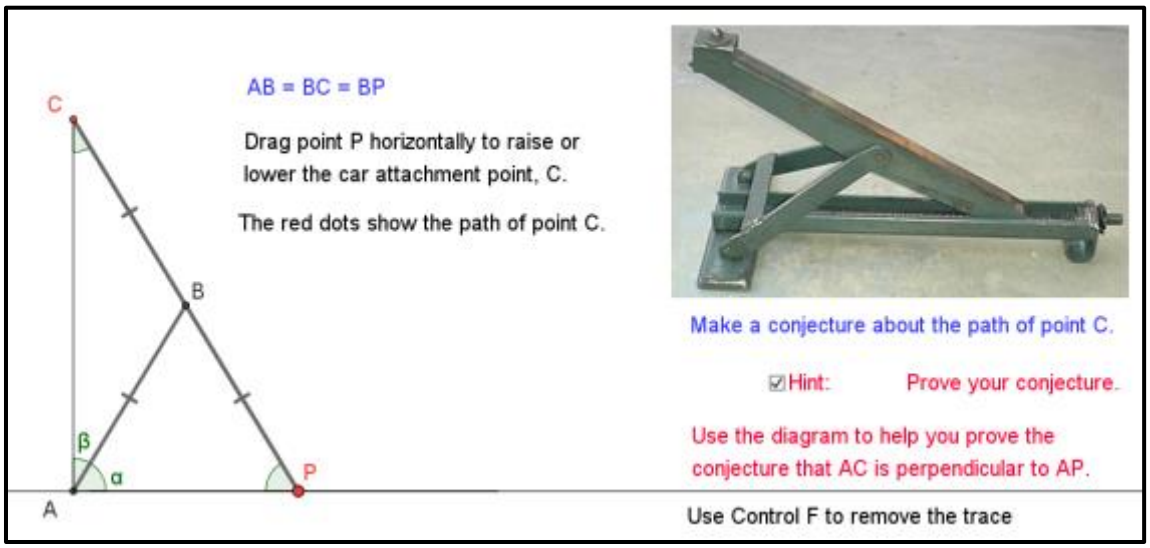
Making a physical model

- Hand out materials for students to construct the physical model. Geo Strips can be used or strips made from card or corflute etc.
- Hand out Student Sheet 1 – Scott Russell Linkage which gives detailed instructions for making the linkage.
- As one student moves point P backwards and forwards, a second student places a pencil in the hole at point C to trace the path of C.
- Ask students to make conjectures about the behaviour of the linkage, and the angles involved. For example: as P moves closer to A, C appears to move upwards at right angles to AP.



Using the computer simulation

- The analysis of the linkage should now become more explicitly geometric.
- Tracing the path of point C should lead to the conjecture that $\angle CAP = 90^\circ$.



Proving the conjecture

- Using the checkbox hint provided in the GeoGebra simulation should assist students to explain why the linkage works. The construction line *AC* is shown and relevant angles are labelled.
- Most students should be able to use the properties of isosceles triangles to label the angle at *P* as α and the angle at *C* as *B*, so in ΔCAP:

  $2\alpha + 2\beta = 180°$ and hence $\alpha + \beta = 90°$
- There are several different proofs. Some samples from students are shown in Teacher Sheet 1 - Scott Russell Linkage Proofs.

Conclusion

The Scott Russell linkage is a useful linkage because it can produce an output movement exactly at right angles to an input movement. The geometry is simple and is based on two isosceles triangles with equal side lengths within a third triangle that can be shown to be a right-angled triangle.

Further Activities

Demonstrate the link with Thales Theorem.

- Thales theorem says that the angle subtended by the diameter of a circle is a right angle.
- A circle can be drawn with centre *B*, passing through *A*, *C* and *P* (all of which are equidistant from *B*). Since *AP* is a straight line segment, passing through the centre, it is a diameter. The angle in a semi-circle is a right angle.
- This fits well with the Australian Curriculum content for level 10A (ACMMG272).

Construct your own simulation

- Students require some experience of dynamic geometry to do this.
- Teacher Sheet 2: Scott Russell Linkage Simulation shows one method.

Investigate other uses of this linkage

- There is a lot of information available on the internet about this linkage and how it is used.

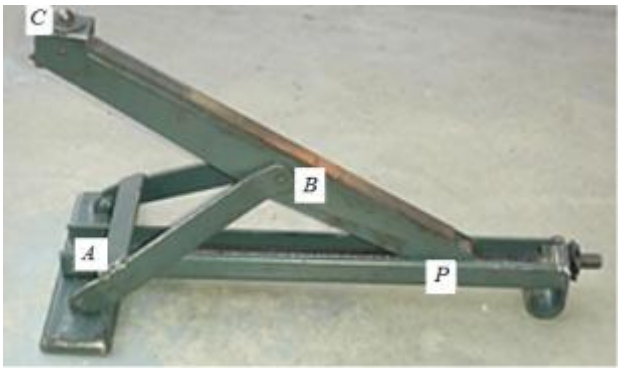
Scott Russell Linkage

Name: _____

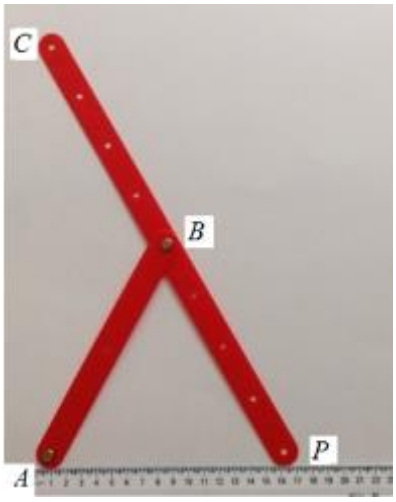
The Scott Russell linkage was originally invented in 1803 by English watchmaker William Freemantle. John Scott Russell was a Scottish naval architect.

An example of the linkage is a type of car jack shown below. This car jack was supplied in the 1970's with some cars, including the Ford Cortina and Lotus Elan. The top of the jack is placed underneath the wheel axle and the horizontal screw (far right) is turned to raise or lower the jack.

The linkage is constructed so that *AB*=*BC*=*BP*. *AP* changes when the screw is turned, which makes point *C* rise.



Make a model



- You will need a long Geo Strip and another one half the length, for example a long red strip and a red strip half the length.
- Point *A* should be fastened to a piece of card with a hole in the card just large enough so that *AB* is free to pivot about point *A*.
- A ruler placed below points *A* and *P* provides a track for *P* to slide backwards and forwards to simulate the screw thread of the car jack.
- While one person slides *P* backwards and forwards along the ruler, the other places a pencil in the hole at point *C*, allowing the pencil to draw the movement of point *C*.
- Write down a conjecture about the movement of point *C* and also a conjectures about the angles that are important.

Observe the simulation and find a proof

- Open the GeoGebra file *Scott Russell*.
- Drag point *P* backwards and forwards and observe the path of *C*.
- Are you still satisfied with your conjectures?
- Use the geometry of the linkage to prove your conjectures.
- How does your finding contribute to the safe operation of the car jack?

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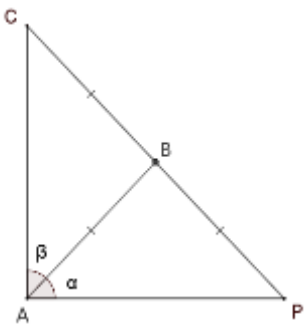
Teacher Sheet 1 - Scott Russell Linkage Proofs



Let $\angle PAB = \alpha$ and $\angle CAB = \beta$

$\angle BPA = \alpha$ (ΔABP is isosceles)

$\angle ACB = \beta$ (ΔABC is isosceles)

In ΔACP,

$\angle APC + \angle ACP + \angle CAP = 180°$

$\alpha + \beta + \alpha + \beta = 180°$

$2\alpha + 2\beta = 180°$

$\alpha + \beta = 90°$

So $\angle CAP = 90°$

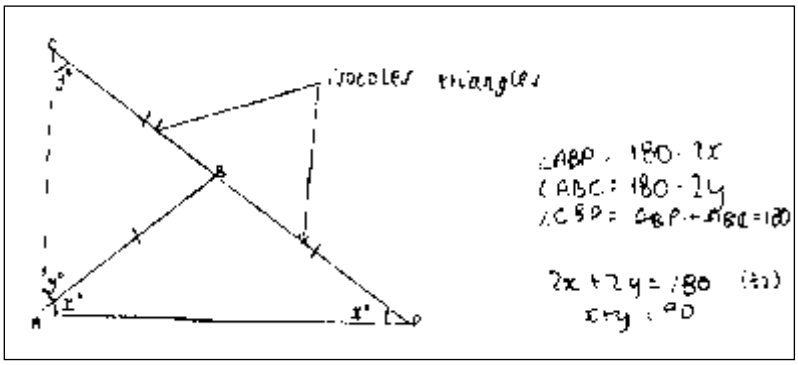
Connection with Thales theorem

Students could explore Thales theorem and develop a proof: the angle in a semicircle is a right angle (i.e. the angle subtended by a diameter of a circle is a right angle). In this case, the three equal bars *AB*, *BC* and *BP* are represented by radii of the circle. The geometry is essentially the same.

Students' conjecturing and proving

In a trial study, students were unsure whether point *P* moved so that *AP* was exactly or only approximately at right angles to *AC*. Using a GeoGebra simulation of the linkage, the students were able to measure the angles, supporting their conjecture for the perpendicular movement. Even though the GeoGebra simulation provided very convincing feedback, the need for an argument based on the geometry of the linkage was stressed. The students accepted this as a challenge.

One pair of Year 8 students constructed the following preliminary argument. After a tentative start, the students drew a labelled diagram and used the exterior angles $\angle ABC$ and $\angle ABP$ of the two isosceles triangles, recognising that these two exterior angles formed a straight angle $\angle CBP$. Dividing both sides of their equation by 2 gave them their proof that $x+y=90°$ and hence that *AC* was perpendicular to *AP*.



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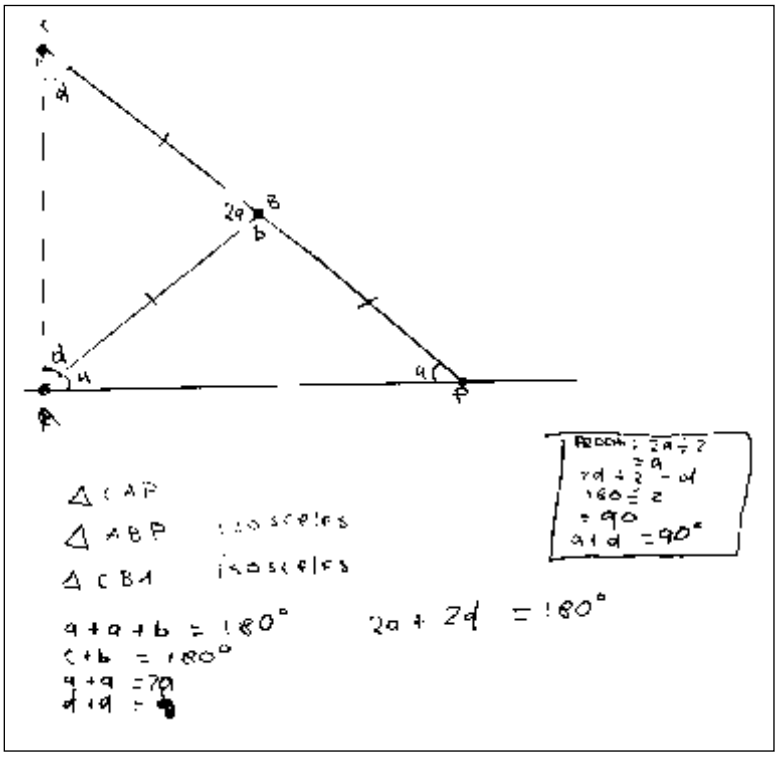
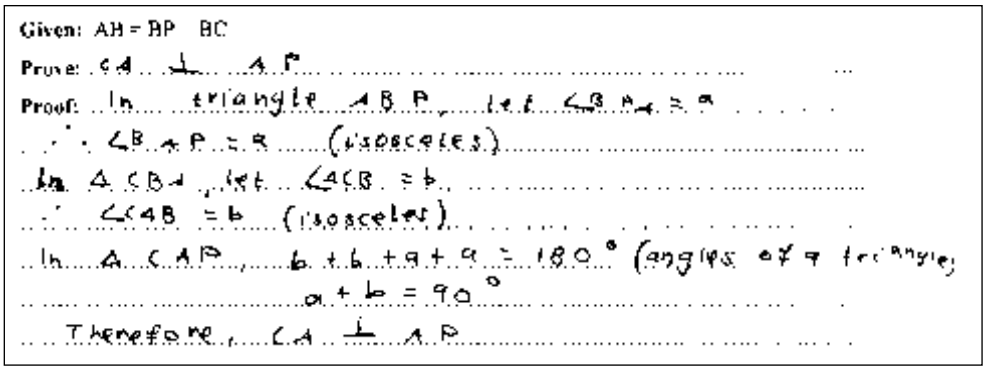
Another pair of year 8 students reached the same conclusion by using angles in the two isosceles triangles and the combined triangle *CAP*.

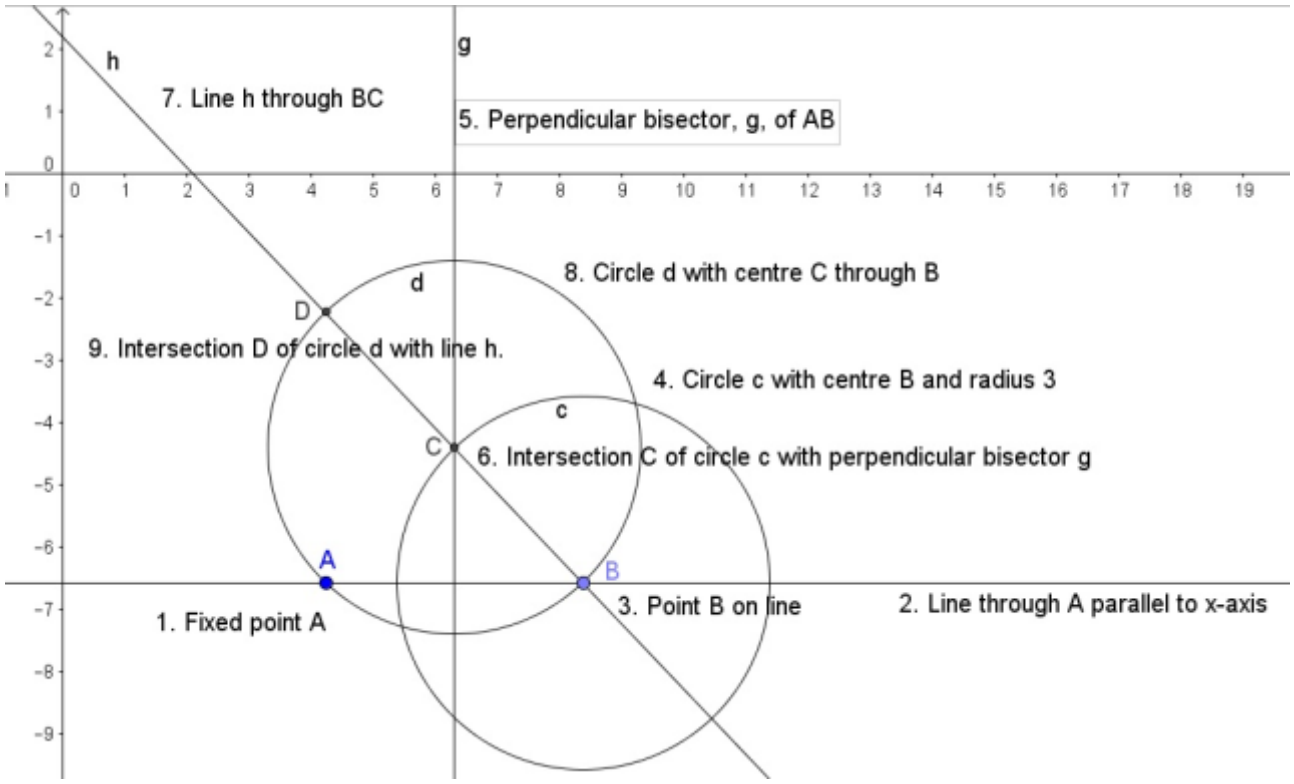


Their tentative reasoning resulted in the construction of the simple 'proof'.



Teacher Sheet 2 – Scott Russell Linkage Simulation

Students who are competent users of dynamic geometry software may construct a simulation of the linkage themselves. A suggested construction is shown below (although points are labelled differently). An arbitrary length of 3 has been chosen for *BC*. The perpendicular bisector of *AB* ensures that *AC* is also 3 units. The circle with centre *C* passing through *B* is used to produce the point *D* so that *CD* is also 3 units. The basic construction can then be refined to hide construction lines and circles, with segments to show the actual linkage. Constructing a working simulation of the linkage involves reasoning about the geometry but also about which points must be fixed, which points must be free to move and which segments must be able to pivot about a point.



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