re(Solve) MATHS BY INQUIRY

MECHANICAL LINKAGES and DEDUCTIVE GEOMETRY

Year 9: Similar Triangles

Lesson 2: The Pantograph

Australian Curriculum: Mathematics (Year 9)

ACMMG220: Use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar.

- Establishing the conditions for similarity of two triangles and comparing this to the conditions for congruence.
- Using the properties of similarity and ratio, and correct mathematical notation and language, to solve problems involving enlargement (for example, scale diagrams).
- Using the enlargement transformation to establish similarity, understanding that similarity and congruence help describe relationships between geometrical shapes and are important elements of reasoning and proof.

Lesson abstract

Students construct a physical model of an enlarging pantograph and use a computer simulation to explore how the copied image compares with the original drawing. They use their knowledge of parallelogram properties and similar triangles to explain how the pantograph works. By investigating pantographs further, students can extend their understanding of scale factors, generalise earlier findings and can design their own pantographs.

Mathematical purpose (for students)

Simple geometry is used in the design of the pantograph, a device to enlarge or reduce drawings.

Mathematical purpose (for teachers)

The lesson provides a novel but challenging way to engage students in geometric deductive reasoning based on parallelogram properties and similar triangles. The linkage provides opportunities for showing points are collinear, showing triangles are similar, and connecting scale factors with similarity. The pantograph can be used in several ways, and the design can be modified in several ways; this provides opportunities for creative work and also for revisiting and generalising the original reasoning.

Lesson Length 100 minutes approximately

Vocabulary Encountered

- pantograph
- similar
- collinear
- scale factor

Lesson Materials

- For each pair of students: 2 pairs of strips with one strip in each pair half the length of the other strip, 5 paper fasteners, sheet of A3 paper
- Slide show: *ST1_Yr9_2a_PantographImages.pptx*
- GeoGebra file *ST1_Yr9_2b_Pantograph.ggb* OR GeoGebra Tube https://ggbm.at/G3pxqept
- GeoGebra file *ST1_Yr9_2c_PantographpivotC.ggb* OR GeoGebra Tube https://ggbm.at/YXTw4uDq
- Student Sheet 1 - Pantograph 1 (1 per student)
- Student Sheet 2 - Pantograph 2 (1 per student where appropriate)

We value your feedback after these lessons via https://www.surveymonkey.com/r/2JH6Z82

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science

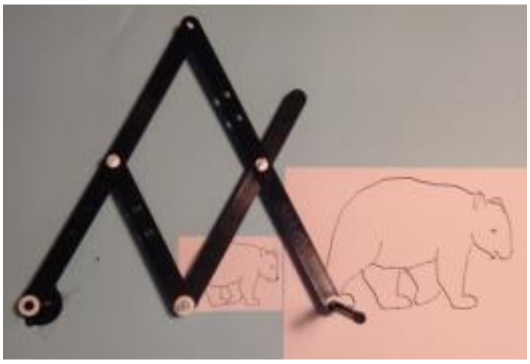
aamt

Introducing the Pantograph

Before the invention of photocopiers, when artists and designers needed to produce exact copies of a drawing, or to enlarge or reduce a drawing, they would use a device called a *pantograph*. Pantographs consist of a set of hinged metal or wooden bars with a pointer to trace around the drawing to be copied, and a pencil that draws the copy.

Mechanical pantographs are no longer in common use in large commercial settings, because they have been replaced by software tools. However, some are still in use in some small operations. For example, the YouTube video at https://youtu.be/iUGkroZus_Y demonstrates how a pantograph is used in woodcarving. A 3D pantograph can be used to copy, enlarge or reduce sculptures.

Pantographs are still available as toys or novelty drawing instruments, for example, the Sketch-a-Graph pantograph is shown below with an enlarged drawing.



Pantographs are still available as toys or novelty drawing instruments, for example, the Sketch-a-Graph pantograph is shown below with an enlarged drawing.

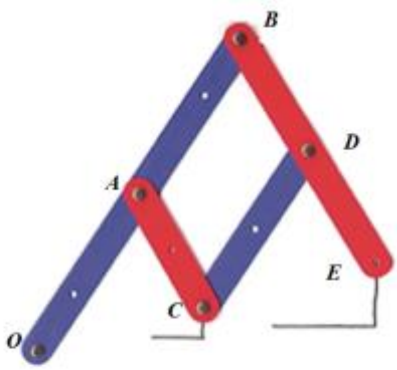
This image, and all others in this lesson plan, are available in the slide show *ST1_Yr9_2a_PantographImages.pptx*

Getting started

- Show students a pantograph if you have one, and illustrate its use, or use the image from the slideshow "Enlarging Pantograph" (ST1_Yr9_2a_PantographIMAGES.pptx).
- Discuss when images might need to be enlarged or reduced, and note that this was difficult in the past.
- Hand out Student Sheet 1 – Pantograph 1 and materials (Geo Strips or home made cardboard or corflute strips, paper fasteners, A3 paper) and ensure all students have access to the computer file.

Models of the Pantograph

Making a physical model



- This pantograph can be constructed easily from plastic Geo Strips and paper fasteners, with $OA=AB=CD$ and $AC=BD=DE$.

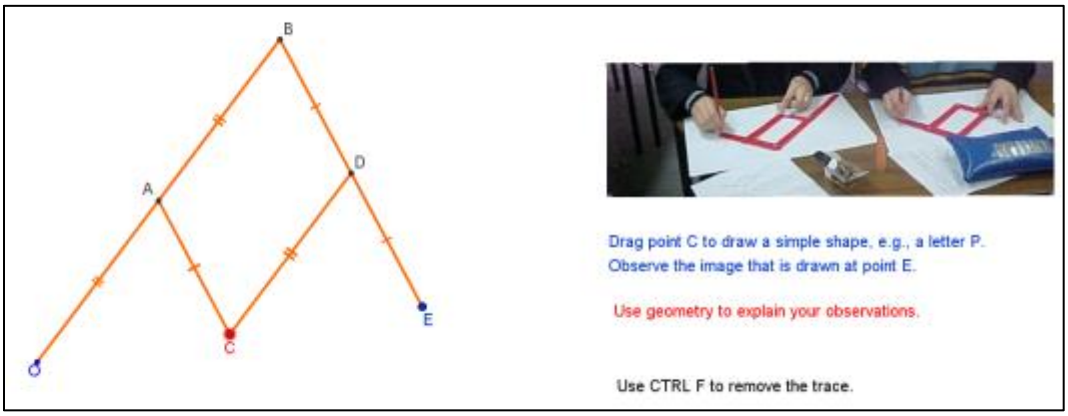
- Point *O* is a fixed pivot point fixed on one corner of a piece of A3 paper. Point *C* can move independently to trace around a drawing but *E* is constrained to follow a path so that it traces out a similar shape to the shape traced by *C*.
- Set students to work in pairs. One student draws a simple picture on the paper near point *C*. This student then traces around the picture by moving point *C* while the other student lightly holds a pencil at point *E* so the pencil traces the enlarged image of the picture drawn at *C*.
- The geometry of the pantograph means that wherever *C* moves, the distance *OE* is always twice the distance *OC*, so the picture at *E* is enlarged by a factor of 2.
- Discuss students' observations.

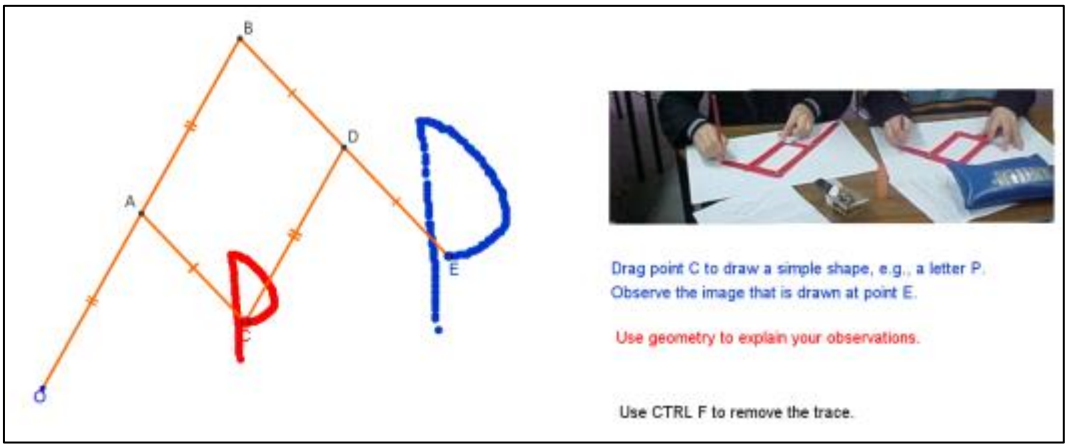
Expected Student Responses

- The image at *E* is larger than the picture at *C*.
- The image at *E* is exactly the same shape as the picture at *C*. (Formally, they are 'similar'.)
- The image at *E* is is about twice as big as the picture at *C*.

Using a computer simulation

The computer simulation allows students to move point *C* to draw a shape and observe the image drawn at *E*.





Explaining How the Pantograph Works

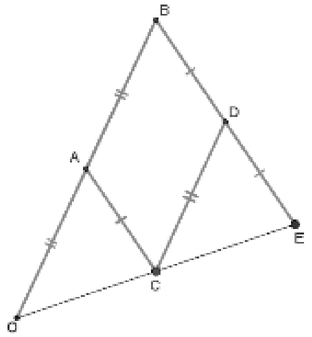
The geometry of the linkage means that *O*, *C* and *E* are collinear. For most students, this can be accepted as a feature of the construction (however, if desired, this collinearity can be proved as in Teacher Sheet 1 – Proofs for Pantographs.

The enlargement factor of 2 can be confirmed by proving that triangles *OAC* and *OBE* are similar (two pairs of sides in same ratio and included angles equal) and hence proving that *OE*=2*OC*. As the pantograph pivots about point *O*, point *E* moves so that its distance from *O* is always twice the distance of point *C* from *O*. This is also an instance of the 'midpoint theorem' for triangles. Details of the proof are given in Teacher Sheet 1 – Proofs for Pantographs.

Structuring the proof

Some students may be able to construct the proof that *OE=2OC* without assistance. Others may be assisted by this series of questions.

- What special quadrilateral is *ABCD?* Explain why.
  - You know that *AB=CD* and *AC=BD*.
  - What else can you say about *AB* and *CD*, and about *AC* and *BD*?
- What do you know about angles *OAC* and *OBE*? Explain.
- What do you know about angles *ACO* and *BEO*? Explain.
- What can you now say about triangles *OAC* and *OBE*? Explain.
- Can you now explain your observation about the image formed at point *E*?



Conclusion

The enlarging pantograph depends on similar triangles in its design. The scale factor is determined by the ratio of sides in the similar triangles.

Further Investigations

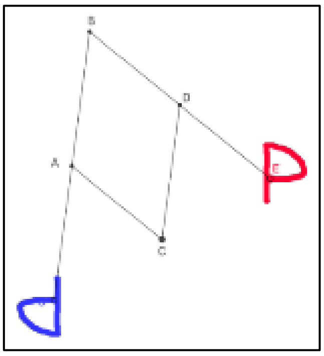
Designing Pantographs with other scale factors

Student Sheet 1 - Pantograph 1 suggests that students investigate:

- How to make a reducing pantograph (by putting the original picture at E, rather than C).
- Designing pantographs with other enlargement factors (3 and 4).

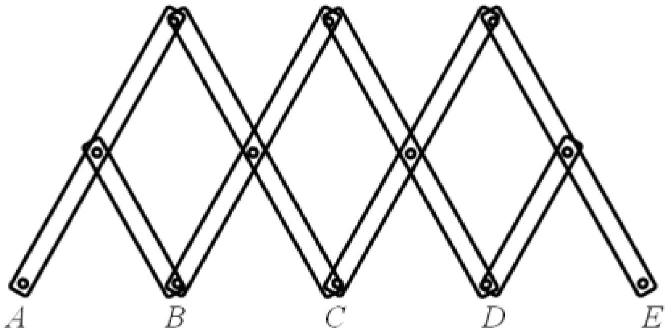
Pantograph with central fixed pivot point

- Students investigate what happens if *C* is the fixed point so that *O* follows *E*? (See Student Sheet 2 - Pantograph 2)
- Students could conjecture then test their conjectures with a Geo Strip model.
- They will find that the image drawn at *O* is the same size as the shape drawn at *E* but rotated through 180°. Points *O* and *E* are turning in opposite directions but they are always the same distance from point *C*.
- The GeoGebra file *ST1_Yr9_2c_PantographpivotC.ggb* (accessible from https://ggbm.at/YXTw4uDq) shows this clearly.
- Students with some experience of dynamic geometry software could make their own computer simulation.



An expanded pantograph

- In the pantograph shown below and on Student Sheet 2 - Pantograph 2, different scale factors can be achieved by varying the position of the fixed pivot point, the tracing point and the image point (where the pencil is placed). Some examples are shown below. Students can investigate how the similar triangles operate in this linkage. They could complete a partly completed table.



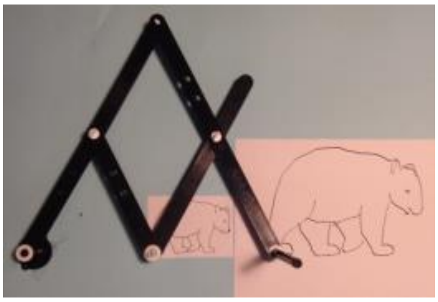
Pivot point	Object point (tracer)	Image point (pencil)	Scale factor
A	*B*	*E*	4
A	*E*	*B*	$\frac{1}{4}$
A	*C*	*D*	1.5
A	*D*	*C*	$\frac{2}{3}$
A	*D*	*E*	1.33
B	*C*	*E*	3

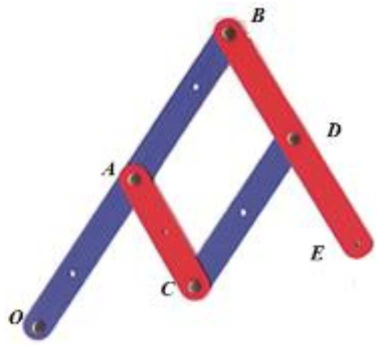
.

Pantograph 1

Name: ____________

Before the invention of photocopiers, when artists and designers needed to produce exact copies of a drawing, or to enlarge or reduce a drawing, they would use a device called a *pantograph*. Pantographs consist of a set of hinged metal or wooden bars with a pointer to trace around the drawing to be copied, and a pencil that traces out the copy.





Making a Physical Model

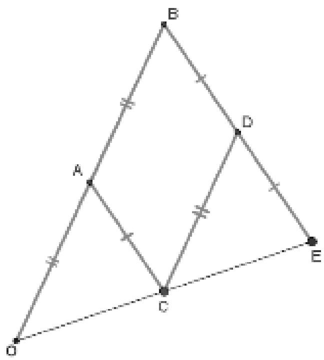
- Assemble the pantograph model as shown. The paper fastener at *O* should be fastened near the edge of a piece of A3 paper.
- Draw a clear shape such as a letter P (about 5 cm high) on the A3 paper near point *C*. This is to be traced.
- Place a pencil in the hole at *E* and allow it to follow the movement of *C* as *C* traces over the drawn shape.
- What do you notice about the image?

Using a Computer Simulation

- Open the GeoGebra file *Pantograph*. This shows a simulation of an enlarging pantograph.
- Move point *C* to draw a shape and observe the image drawn at *E*.
- Make a conjecture about the size of the image at *E*.

Explaining How the Pantograph Works

- Using the following diagram and your understanding of similar triangles, explain why the enlarging pantograph produces an image twice the size of the original drawing.



Modifying the Pantograph

- What do you think might happen if the original shape was drawn at *E* instead of *C* and the pencil was placed at *C* to make the copy?
- Design and make a physical model of a pantograph with enlargement of 3, and test it.
- Design and make a physical model of a pantograph with enlargement factor of 4, and test it.

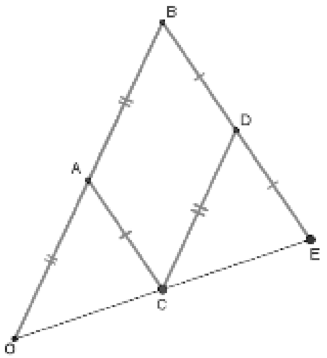
Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

re(Solve) MATHS BY INQUIRY

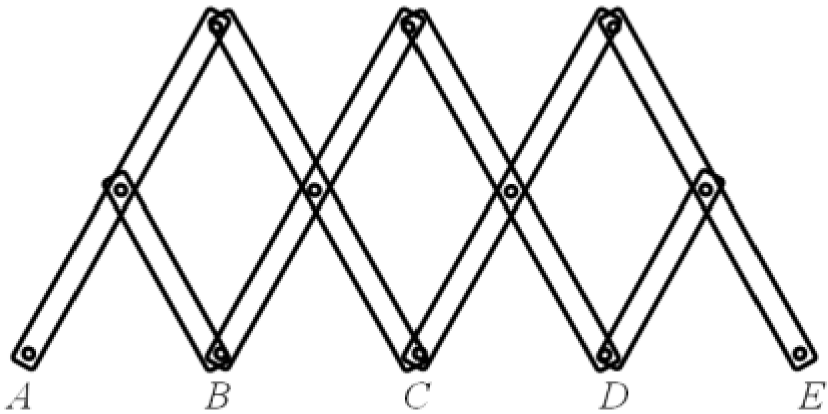
Pantograph 2

Name: ______________

1. What would happen if *C* was the fixed point so that *O* followed *E*? Test your conjecture with a physical model. Can you explain what happens?



2. In the following pantograph, different scale factors can be achieved by varying the position of the fixed pivot point, the tracing point and the image point (where the pencil is placed).



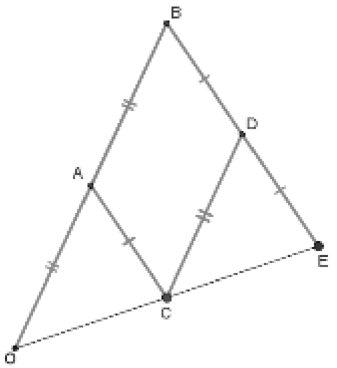
Some examples are shown below. Complete the table.

Pivot point	Object point (tracer)	Image point (pencil)	Scale factor
A	*B*	*E*	4
A	*E*	*B*	
A	*C*		1.5
A			$\frac{2}{3}$
A	*D*	*E*	
	C		3

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training

Australian Academy of Science
aamt

Teacher Sheet 1 – Proofs for Pantographs



Proof that O, C and E are collinear (optional)

AB is parallel to CD so OB is parallel to CD (A is a point on line OB)
So $\angle OAC = \angle ACD$ (alternate angles between parallel segments cut by transversal AC)
AC is parallel to BD so AC is parallel to BE ($ABCD$ is a parallelogram)
So $\angle AOC = \angle DCE$ (corresponding angles)
So $\angle ACO + \angle ACD + \angle DCE = \angle ACO + \angle OAC + \angle AOC = 180°$
So OCE is a straight angle and O, C and E are collinear

Proof that $OE = 2OC$

In ΔOAC and ΔOBE
$\angle AOC$ is common
$\angle OAC = \angle OBE$ (corresponding angles)
$\angle ACO = \angle BEO$ (corresponding angles)
So ΔOAC and ΔOBE are similar triangles (A.A.A.)
$OB = 2OA$ (given)
So $OE = 2OC$

Hence whatever distance C is from O, E is always twice the distance away. The image is therefore twice the size.

Mathematics by Inquiry is an initiative of, and funded by, the Australian Government Department of Education and Training