re(Solve) MATHS BY INQUIRY

MECHANICAL LINKAGES and DEDUCTIVE GEOMETRY

Teachers' Guide

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Mechanical Linkages and Learning Geometry

What are mechanical linkages?

The operation of many everyday items—such as folding ironing tables, umbrellas, car jacks and scissor lifts—involves a set of hinged bars which can slide along a path or rotate about each other or about fixed pivot points according to the geometry underlying their construction. Sets of hinged bars of this type are called *mechanical linkages*.

While most mechanical linkages are complex and their designs are beyond school geometry, the carefully selected linkages in these lessons offer a wealth of geometry appropriate for secondary school mathematics. Incorporating simple geometry, for example based on parallelograms and kites, many everyday items provide an ideal real-world context for students in the middle years to become acquainted with special quadrilateral properties.

Drawing instruments (pantographs) based on mechanical linkages were widely used in the 16th and 17th centuries for enlarging or copying drawings and for perspective drawing. Simple enlarging pantographs are still produced as novelty drawing toys, or used in specialist tools for hobbyists. The design of these is based on similar triangles.

We value your feedback after these lessons via https://www.surveymonkey.com/r/2JH6Z82

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Following the invention of the steam engine in the 18th century, there was a challenge to design linkages to convert between linear movement and circular movement (e.g. from the engine's piston to turn a wheel or shaft). Several 19th-century mathematicians took up the challenge. Many of the linkages they designed converted circular movement into approximately linear movement, but this was adequate for many purposes. However, there was an ongoing challenge to design linkages that would produce exact linear motion. Many of these were of theoretical interest only because the more complex a linkage, the more wear and tear there is on moving parts. Exploring some of these linkage inventions provides a context for geometric reasoning incorporating similar and congruent triangles, isosceles triangles, and circle geometry for Years 9 and 10. There is great opportunity to extend to the goals of the Australian Curriculum 10A.

Rationale

"I was good at maths at school but I hated geometry – all we ever did was calculate the third angle in a triangle": this comment from a practising Australian teacher perhaps highlights why there is a need for motivating contexts in which our students can explore geometry. When devising these tasks in deductive reasoning in geometry, the guiding principles have been:

- Choosing geometry appropriate to the students' level
- Providing rich visual imagery, both static and dynamic
- Providing an opportunity for students to use the language of geometry
- Providing meaningful context that can motivate argumentation and conjecturing
- Establishing the students' need for deductive reasoning as an answer to the question 'why'
- Providing an opportunity for students to identify mathematical features (e.g. angles, lines) in real world objects through a process of abstraction
- Providing links with other STEM subjects, in particular, engineering and technology
- Providing links with history through historical inventions

Goldenberg, Cuoco and Mark (1998) assert that:

> *For some students, a visual approach may be absolutely essential. For such students, including many who consider themselves to be poor at mathematics, visual approaches are access. Thus, for many students, visualization and visual thinking serve not only as a potential hook, but also as the first opportunity to participate.*

Ideally the deductive reasoning in these lessons is supported by students' seeing the movement of real linkages, physical models, and computer simulation.

The motivating environment of mechanical linkages can be exploited to prompt students to question why each linkage works the way it does. In the words of one year 8 girl working with a car jack: "We think it's a right angle but now we have to work out why". Geometric reasoning then becomes an answer to the question 'Why'? Whilst formal proofs are not an expected outcome of the lessons, at Year 8 level the slideshows *Rhombus properties*, *Kite properties* and *Is it a parallelogram*? can be used as the basis for class discussions involving the development of logical sequences of geometric statements. In this way, the seeds of deductive reasoning are being sown. At years 9 and 10 it would be hoped that some students would be able to develop sequences of reasoning by themselves, although some may require prompting questions on which to base their reasoning. Lesson plans include such prompts for the more difficult linkages.

In a typical classroom, geometric properties are frequently taught as facts, followed by practise where, for example, students' calculate unknown angles. In the Australian Curriculum, however, there is an expectation that students will engage in reasoning beyond simple angle finding. For example, at Year 8, the Australian Curriculum includes the following:

ACMMG202: Establish properties of quadrilaterals using congruent triangles and angle properties.

- Establishing the properties of squares, rectangles, parallelograms, rhombuses, trapeziums and kites.
- Identifying properties related to side lengths, parallel sides, angles, diagonals and symmetry.

The sequence of lessons *Mechanical linkages and deductive reasoning* has been designed to provide a motivating real-world context for students to engage in deductive reasoning in geometry in line with the relevant content prescribed by the Australian Curriculum.

Students use both linkages and models

A strong feature of these lessons is that students are able to see and manipulate the linkages. Students operate the real linkage whenever possible, they make physical models out of plastic strips or light card, and they use pre-prepared dynamic geometry computer simulations.

Whilst the tactile experience of operating the actual tool or a physical model of the linkage provides an instant sense of satisfaction, further visual exploration can be achieved with computer simulations using dynamic geometry software such as GeoGebra. Students can observe what stays the same and what varies as the dynamic geometry models are operated.

Using real examples of the linkages

For some students the benefits in engagement from having physical examples of the mechanical linkages are significant. The tactile feel of operating these objects or their models has been found to provide a very motivating context for students. Consider, for example, the disengaged 14-year-old boy whose interest in geometry was suddenly captured when he made a rhombus linkage from plastic geometry strips: “My Dad drives one of those scissor lifts” as he played with the linkage, or the 13-year-old girl, when working with the *Consul* (toy calculator) linkage exclaimed: “Seven squared, oh yeah! Forty-nine! This is cool! I want one of these!”

Wherever possible, it is a good idea to have actual examples of the linkages, for example, a cantilever toolbox, a car jack, a folding umbrella. In the case of linkages such as a scissor lift and a cherry picker, images will have to suffice unless you are lucky enough to have an example on an on-site building construction. The slideshows include images of the linkages relevant to each lesson. A list of suppliers to purchase linkages (current at time of publication) is given below.

Making physical models from plastic strips or light card

Construction of the linkages with plastic strips, e.g. commercially-produced Geo Strips, is the desirable option. The Geo Strips are sufficiently rigid to mimic the movement of the linkage in the real-world object. The models are quick to assemble, particularly if the slideshow image of the physical model is shown to students as they assemble their model. These plastic strips also allow students to modify designs of certain linkages (e.g. enlarging pantograph), to create a new tool with similar geometry, e.g., a different enlargement factor. One drawback is the propensity for paper fasteners (split pins) to end up on the floor!

There are alternatives if Geo Strips are not available. Cutting strips from card using the templates provided in some lessons is an option, as is making them from materials such as Corflute. This takes more time. Light card is not sufficiently rigid for some linkages. Once again, it should be kept in mind that constructing the model is not the ultimate goal of the lesson – there must be time to discuss the underlying geometric properties. The card templates provided with several lessons require scissors but no paper fasteners. These models work satisfactorily but not as well as Geo Strip models.

Geo Strips are available from Modern Teaching Aids, Sydney https://www.teaching.com.au/

Computer simulations

The unique capabilities of dynamic geometry software (e.g. GeoGebra) permit the construction of computer models which simulate the behaviour of the actual linkages, but at the same time represent them as geometric figures. In contrast to the chunky physical model, the dynamic geometry construction resembles a geometric drawing. It is also possible to get accurate measurements of lengths and angles. It should be noted, though, that the construction of these computer models is not straightforward as certain points and lines must be constructed in the correct order so that the model accurately mimics the behaviour of the physical linkage. For this reason prepared simulations are provided with each lesson.

The simulations accompanying these lessons are presented as GeoGebra files. GeoGebra software is free to download, or can be used directly from the web without any installation. Both options are supported in the lesson notes. No knowledge of this software is required to use the files for the main part of any lesson – the only actions are dragging points and checking check boxes. Modifications to files can only be made if the software has been downloaded.

A note of caution. The accuracy of angle and distance measurements in dynamic geometry software tempts us into using such measurements as proof of conjectures based on our observations. However, despite the software’s accuracy and its ability to provide very convincing visual evidence, it is important that students appreciate that it is just that: strong visual evidence to give us confidence in our conjectures. Deductive reasoning based on a logical sequence of geometric statements is the ultimate goal in these lessons, particularly at Years 9 and 10.

Overview of Units and Lessons

The lessons are grouped into four units according to how the geometry fits with the Australian Curriculum content:

- Angles and lines (Australian Curriculum Outcomes for Year 7)
- Quadrilateral linkages (Australian Curriculum Outcomes for Year 8)
- Similar triangles (Australian Curriculum Outcomes for Year 9)
- Proof (Australian Curriculum Outcomes for Year 10 and 10A)

Curriculum links

Lessons	**Australian Curriculum content**
Angles and lines Folding quadrilaterals; An extended protractor	**ACMMG164**: Investigate conditions for two lines to be parallel and solve simple numerical problems using reasoning. • Defining and identifying the relationships between alternate, corresponding and co-interior angles for a pair of parallel lines cut by a transversal. **ACMMG165**: Classify triangles according to their side and angle properties and describe quadrilaterals. • Describing squares, rectangles, rhombuses, parallelograms, kites and trapeziums.
Quadrilateral linkages Folding umbrella, Toolbox, Car jack, Scissor lift, Cherry picker, Angle bisectors	**ACMMG201**: Develop the conditions for congruence of triangles. • Solving problems using the properties of congruent figures. **ACMMG202**: Establish properties of quadrilaterals using congruent triangles and angle properties. • Establishing the properties of squares, rectangles, parallelograms, rhombuses, trapeziums and kites. • Identifying properties related to side lengths, parallel sides, angles, diagonals and symmetry.
Similar triangles Ironing table, Enlarging pantograph	**ACMMG220**: Use the enlargement transformation to explain similarity and develop the conditions for triangles to be similar. • Establishing the conditions for similarity of two triangles and comparing this to the conditions for congruence. • Using the properties of similarity and ratio, and correct mathematical notation and language, to solve problems involving enlargement (for example, scale diagrams). • Using the enlargement transformation to establish similarity, understanding that similarity and congruence help describe relationships between geometrical shapes and are important elements of reasoning and proof.
Proof Chebycheff's linkage, Scott Russell car jack, Pascal's angle machine, Consul, Sylvester's pantograph, Peaucellier's linkage	**ACMMG243**: Formulate proofs involving congruent triangles and angle properties. • Applying an understanding of relationships to deduce properties of geometric figures (for example the base angles of an isosceles triangle are equal). **ACMMG244**: Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes. • Performing a sequence of steps to determine an unknown angle giving a justification in moving from one step to the next. • Communicating a proof using a sequence of logically connected statements. **ACMMG272**: Prove and apply angle and chord properties of circles. • Performing a sequence of steps to determine an unknown angle or length in a diagram involving a circle, or circles, giving a justification in moving from one step to the next.

Advice on lesson selection

Introductory task

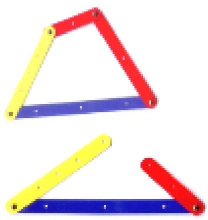
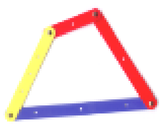
It is recommended that the short introductory task is included at the beginning of each unit where classes are doing the Mechanical Linkage lessons for the first time. It provides an introduction to the concepts of the triangle being a rigid shape while the hinged quadrilateral is a flexible shape. Students gain an understanding of what is meant by a mechanical linkage when they see how the shape of the Geo Strip quadrilateral can be changed.

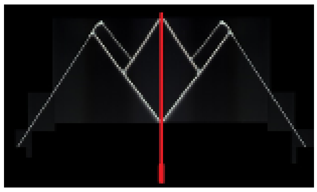
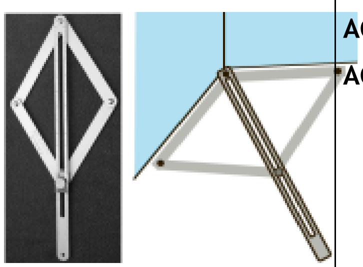
An important understanding that is incorporated into the lesson is that not all side lengths will make a triangle. Given a random set of three Geo Strips, some can make a triangle whilst others cannot. Discussion brings out the concept that the lengths of the two shorter sides of the triangle together must add to more than the length of the third side and students can appreciate why this is.

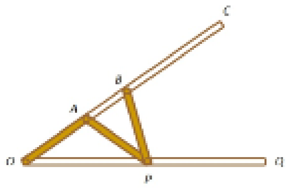
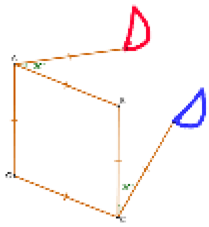
Common underlying geometry provides choice

It is not necessary for students to do all of these lessons. Several lessons involve the same underlying geometry. For example, the Year 8 lessons on the scissor lift and the car jack both depend on the rhombus properties of equal sides and the perpendicular diagonals. The folding umbrella, the cherry picker and toolbox all depend on the fact that if a quadrilateral is constructed with opposite sides equal, then it will be a parallelogram. Teachers might choose only one of the linkages, or may have different groups in the class working on different linkages.

List of lessons

Lesson	Brief Description		Australian Curriculum Links
Introductory Task **(recommended for all year levels; supplied at the end of this document)**			
Introductory task	A triangle is a rigid shape, but a quadrilateral is not. A linkage has fixed side lengths (nearly always), but the angles can change. Familiarises students with the construction materials and gives examples of mechanical linkages.		
Angles and Lines (recommended for Year 7)			
1. Folding Quadrilaterals	This lesson can be used as an alternative to the introductory task. Students explore the lengths of bars that will make triangles and quadrilaterals, and investigate which quadrilaterals will fold down neatly.		**ACMMG165**
2. An extended protractor	Students explore the design of a protractor that has a rhombus linkage attached to it. They identify relationships between angles associated with parallel lines and explain how the protractor works.		**ACMMG164** **ACMMG165**
Quadrilateral Linkages (recommended for Year 8)			
1. Car jack (Can be combined with Scissor lift lesson)	Starting with the definition of a rhombus – a quadrilateral with four equal sides – students learn to reason deductively about other rhombus properties and explore the properties that govern the operation of the jack.		**ACMMG201** **ACMMG202**

2. Scissor lift (Can be combined with Car jack lesson)	Students investigate the design and operation of a scissor lift and the part played by rhombus properties		**ACMMG201** **ACMMG202**
3. Folding umbrella (can be combined with Cherry picker and Toolbox lessons)	Each rib in a folding umbrella includes a hinged quadrilateral with opposite sides equal. Students learn to reason deductively that such a quadrilateral is a parallelogram. The parallel sides ensure that the umbrella folds neatly.		**ACMMG201** **ACMMG202**
4. Cherry picker (can be combined with Folding umbrella and Toolbox lessons)	A linkage constructed with both pairs of opposite sides equal is useful because its opposite sides remain parallel. Combining two or more parallelograms (articulated parallelograms) as in a cherry picker allows for accurate and flexible positioning of the work platform.		**ACMMG201** **ACMMG202**
5. Toolbox (can be combined with Cherry picker and Folding umbrella lessons)	Students investigate the design and operation of a cantilever toolbox. The brackets that connect the trays to each other are attached to form quadrilaterals with equal opposite sides. The quadrilaterals are therefore parallelograms and allow the trays to open out and close while remaining horizontal and parallel to each other.		**ACMMG201** **ACMMG202**
6. Angle bisectors	A rhombus and a kite both have geometric properties that make them useful in angle bisecting tools. In each case a diagonal divides the quadrilateral into two congruent triangles and bisects a pair of opposite angles of the quadrilateral.		**ACMMG201** **ACMMG202**
Similar Triangles (recommended for Year 9)			
1. Ironing tables	When an ironing table with legs that pivot is raised or lowered, the top always stays parallel to the floor. Students investigate the triangles formed by the pivoting legs of three different designs of ironing table, in particular investigating how different leg lengths and pivot positions ensure that similar or congruent triangles are formed.		**ACMMG220**

2. Pantograph	Students investigate an enlarging pantograph and use their knowledge of parallelogram properties and similar triangles to explain how the pantograph works. They can extend their understanding of scale factors, and design their own pantographs.		**ACMMG220**
Proof (recommended for Year 10 and 10A)			
1. Chebycheff's linkage	Students investigate Chebycheff's linkage and determine whether it produces approximate or exact linear motion from circular motion. Visually, the motion appears as a very convincing straight line, so students are misled until measurements expose the small variation from linear motion. This establishes awareness of the need for proof in other geometry contexts.		**ACMMG243** **ACMMG244**
2. Scott Russell car jack	As the horizontal screw of the Scott Russell car jack is turned, the car attachment point moves vertically (perpendicular to the screw). Students use the geometry of the linkage to find if the motion is really vertical. This involves reasoning about angles in three connected triangles.		**ACMMG243** **ACMMG244**
3. Pascal's angle machine	Students investigate the design and operation of an angle trisector invented by mathematician Blaise Pascal, deliberately named here *Pascal's angle machine*. They first explore the angle machine to find out what its purpose is then use the geometry of isosceles triangles and exterior angles to prove why the machine works.		**ACMMG243** **ACMMG244**
4. Consul	Students construct a physical model of Consul, a toy calculator, and use a computer simulation of Consul to explore how the geometric design enables the product of numbers between 2 and 12 to be displayed. Students may then develop a proof based on a sequence of deductive reasoning.		**ACMMG243** **ACMMG244**
5. Sylvester's pantograph	Sylvester's pantograph was designed for copying drawings. Students explore how the copied image compares with the original drawing then develop a geometric proof based on a sequence of deductive reasoning in which they use their knowledge of rhombus properties and congruent triangles.		**ACMMG243** **ACMMG244**
6. Peaucellier's linkage	Peaucellier's linkage converts circular motion to linear motion. Students explore how the linkage moves and discover how the geometric design of the linkage allows it to produce exact linear motion.		**ACMMG272**

Teaching Advice

Flexibility in Lesson Plans

The format of each lesson can be modified by teachers according to class size, ability of students or availability of resources. At Year 8 level, for example, the teacher of a highly motivated and capable class may choose to spend less time on physical model making and move quickly to the computer simulation and deductive reasoning. More than one of the linkages could then be covered in the one lesson, for example, the scissor lift and car jack which are both dependent on the same rhombus properties. On the other hand, other teachers may consider the motivating aspect of creating the physical model to be more important. In either case it would be hoped that there was adequate time for drawing out the rhombus properties that underlie the design of the object. In between these two scenarios, there is room for differential teaching with students proceeding to some extent at their own pace, provided the lesson progresses for all students beyond mere model making.

Productive Group Work

Working in pairs seems to provide all students with the opportunity to participate in model making and productive discussion. However, it should be kept in mind that constructing the model is not the ultimate goal of the lesson.

Using the Slideshows

Each lesson is accompanied by a slide show that includes images that will be useful throughout the lesson.

- It is not intended that the entire slide show is shown at once.
- Key points where particular slides are to be shown are highlighted in the lesson plan. For example, the first few slides introduce students to the linkage then perhaps show a Geo Strip or card strip model.
- After students have had a chance to construct their models, a class discussion of the key features could lead into the computer simulation.
- Students could explore the computer simulation on their own, with the slideshow screen image merely showing students the appearance of the file they should be opening.
- Alternatively, the teacher could choose to demonstrate the computer simulation to the whole class, with students engaging in whole class discussion.
- A whole class summing up of the lesson, including any additional slideshows, such as Rhombus properties, would then ensure that teachers can emphasise the underlying geometry.
- Some lessons include links in the slideshows to online videos. The appropriate time to show these is suggested in the slideshow.

Notes from Trialling

Reading carefully through the lesson plan in advance is helpful. Teachers then become aware of any pitfalls and are clear about the focus of the underlying geometry so they can check that students are not wasting time by making irrelevant measurements.

Sourcing Linkages

Extended protractor	Available on Amazon: General Tools 29 Plastic Protractor and Angle Finder with Articulating Arms
Car jack	Supplied with many cars
Scissor-lift type linkage	Bathroom mirror with extending bracket available from Ikea: FRÄCK mirror Expanding garden trellis (hardware shop).
Folding umbrella	An old umbrella pulled apart provides 8 ribs, each with ‘parallelogram’ linkage
Toolbox	Cantilever metal toolboxes available from major hardware retailers.
Angle bisectors	Rhombus version of angle bisector is available from McJing Tools, NSW. It is listed under Woodworking Tools, Measuring Tools as an Angle Divisor. https://mcjing.com.au/
Enlarging pantographs	Sketch-A-Graph is available from several Australian online stores for approximately $20.
Consul	Available from several online stores, including https://notjustretro.com.au/products/educated-monkey/

	http://aussiebrewer.com.au/shop/gifts/consult-the-educated-monkey/ https://picclick.com.au/Amazing-Monkey-Multiplication-1916-Tin-Toy-Calculator-Collectors-232254125084.html
Geo Strips	Available from Modern Teaching Aids, Sydney https://www.teaching.com.au/

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Authorship

These lessons were produced by Dr Jill Vincent. Over many years, Jill has explored mathematics in the world around us. She created the maths trail "From Shrine to University: A Geometry Journey Along St Kilda Road and Swanston Street" (Mathematical Association of Victoria) which shows the mathematics in architecture and public art on a walk along the main thoroughfare in Melbourne. In her PhD Jill used mechanical linkages, including the Scott-Russell car jack, Pascal's angle machine, the enlarging pantograph, Sylvester's pantograph and Consul, to investigate secondary school students' geometric argumentation.

Additional input from the reSolve team was provided by Kaye Stacey and Lucy Bates.

Introductory Task

Aim

- This task provides a brief introduction to mechanical linkages for students who are undertaking their first lesson in from *Mechanical Linkages and Deductive Geometry*.
- The suggested time is approximately 20 minutes.
- The key points are to see examples of mechanical linkages and to note that triangles are rigid shapes but quadrilaterals are not. Mechanical linkages are based on the fact that quadrilaterals are not rigid.
- Students are also familiarised with the materials used to construct the linkages in other lessons.

Materials required

- At least four Geo Strips per student and four paper fasteners (split pins). Use a variety of lengths.
- If Geo Strips are not available, students can cut strips from cardboard. Make sure that the holes for the paper fasteners are small enough to hold the shapes without too much movement.

What is a mechanical linkage?

Explain some of the key points from the first section of this Teachers' Guide.

The operation of many everyday items—such as folding ironing tables, umbrellas, car jacks and scissor lifts—involves a set of hinged bars which have a fixed length, but are free to move. The lengths of the bars stay the same, but the angles between them can change.

Show some examples and highlight how the bars are of a fixed length but the angles can change:

- real examples are good (e.g. car jack, expanding arm to pull out a mirror, folding umbrella mechanism)
- show the pictures in the slide show *ST1_0_Linkages_Intro_Images.pptx*.

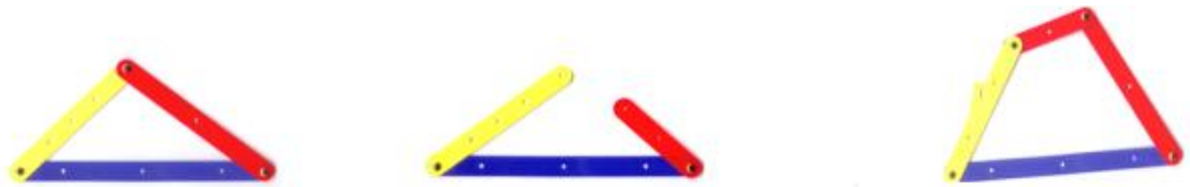
Joining three strips to form a triangle

Students choose three Geo Strips and three paper fasteners to make a triangle.

Some students may have chosen strips that will not make a triangle. Discuss with the class that to make a triangle, the sum of the lengths of the two shorter strips must be greater than the length of the third strip, otherwise the two shorter strips will not meet. These students should change their strips so that they have a triangle.

Students can then observe that their triangle has a fixed shape. It is rigid. This is why triangles are used where strength is required (for example in building bridges).

Ask students who have used strips of the same length to compare their triangles. Highlight the finding that if the side lengths of two triangles are the same, then the triangles will have the same angles. It is not possible to make two different triangles with the same side lengths.



A triangle is a rigid shape. Some bars will not make a triangle. A quadrilateral is not rigid.

Joining four strips to form a quadrilateral

Students choose four Geo Strips and four paper fasteners to make a quadrilateral.

Some students may have chosen strips that will not make a quadrilateral. Discuss with the class that to make a quadrilateral, the sum of the lengths of the three shorter strips must be greater than the length of the longest strip, otherwise the three shorter strips will not meet.

As soon as the quadrilateral has been constructed, students will note that it can change shape. It is not rigid. This is why quadrilaterals are used where movement is required (for example in a car jack).

Ask students who have used strips of the same length to compare their quadrilaterals. Highlight the finding that if the side lengths of two quadrilaterals are the same, then the angles might be different.

Many everyday objects make use of the changing shapes of joined bars like the Geo Strip quadrilateral so they can pivot at the corners. This is the basis of mechanical linkages.