Year 8: Quadrilateral Linkages
Lesson 4: Toolbox

Australian Curriculum: Mathematics (Year 8)

ACMMG201: Develop the conditions for congruence of triangles.
- Solving problems using the properties of congruent figures.

ACMMG202: Establish properties of quadrilaterals using congruent triangles and angle properties.
- Establishing the properties of squares, rectangles, parallelograms, rhombuses, trapeziums and kites.
- Identifying properties related to side lengths, parallel sides, angles, diagonals and symmetry.

Lesson abstract

Students investigate the design and operation of a cantilever toolbox to see how the geometric properties of a parallelogram contribute to how the trays of the toolbox open and close. Students observe an actual toolbox, make a physical model and use a computer simulation to understand how the design based on parallelograms constructed from bars of equal length allows the toolbox to open and close with the trays remaining horizontal. There is an opportunity for deductive reasoning, supported by a slideshow.

Mathematical purpose (for students)

A linkage constructed with both pairs of opposite sides equal is useful because its opposite sides remain parallel.

Mathematical purpose (for teachers)

A cantilever toolbox is designed so that its trays open and close while remaining horizontal. The brackets that connect the trays to each other are attached to form quadrilaterals with equal opposite sides. These quadrilaterals are therefore parallelograms and this ensures the desired movement. The purpose is for students to observe real motion, identify geometric properties visually, and then engage in deductive reasoning about them, proving the quadrilaterals are parallelograms.
The task can be presented to students alongside the *Folding umbrella* activity and the *Cherry picker* activity, because similar mathematical reasoning is involved.

Lesson Length 50-60 minutes (or 120 minutes for Toolbox, Folding Umbrella and Cherry Picker)

Vocabulary Encountered
- horizontal
- vertical
- diagonal
- congruent
- rivet
- bracket

Lesson Materials
- Cantilever toolbox, (e.g. as available from Bunnings, optional)
- Geo Strips or similar (4 short strips and 2 twice as long per pair of students, or print these strips onto card)
- Paper fasteners (14 per pair of students)
- A3 paper or light card (1 sheet per pair of students)
- GeoGebra file *ST1_Yr8_4b_Toolbox.ggb*
- OR GeoGebra Tube link https://ggbm.at/V4eMwGSk
- Slide show: *ST1_Yr8_3c_IsItParallelogram.ppt*
- Slide show: *ST1_Yr8_4a_ToolboxImages.pptx*
- Student Sheet 1 – Toolbox (1 per student)

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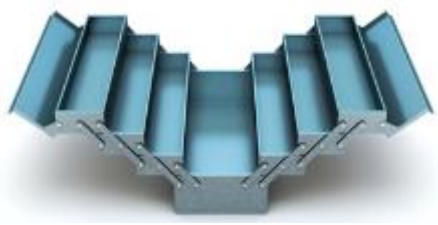
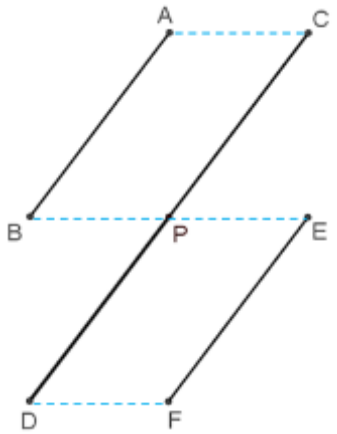
Introducing the Toolbox

Toolboxes as shown below are referred to as cantilever toolboxes because of the way the trays open out. The cantilevered design is commonly used in tool boxes, fishing boxes and sewing boxes. The bars that link the trays are attached so that they form parallelograms. As in the case of the umbrella rib linkage, the design is based on linkages that form parallelograms. One pair of opposite sides of each parallelogram is formed by the equal length bars. The other pair of opposite sides is formed by the equal horizontal spacing of the rivets about which the bars pivot. In this linkage, the bars are kept parallel because quadrilaterals are constructed to have opposite sides equal length.





Showing the parallelograms.

Getting Started

- Show students a toolbox like this (can be purchased from hardware shops) and demonstrate how it opens and closes. Also show the images of toolboxes in the slide show *ST1_Yr8_4a_ToolboxImages.pptx*
- Discuss the useful features of the design.

Expected Student Response

- The trays stay horizontal as the box is opened, so that the contents do not spill.
- The box closes compactly.
- When the toolbox is open, it is easy to get things from any of the trays.

Models of the Toolbox

Making a physical model

Students now make a 2-dimensional model of the toolbox. Doing this highlights important geometric features of the construction. Rectangular pieces of card will represent the trays, paper fasteners will rivet the bars to the trays and Geo Strips will represent the bars. Alternatively, strips can be cut from card.

The first step is to draw the trays on an A3 sheet of paper or card. Divide the sheet into three equal strips, and halve the top two strips (see below). DO NOT CUT the trays. Next, students should observe and discuss carefully the arrangements of the rivets (paper fasteners) and the lengths of the bars. Mark the positions of the paper fasteners, then cut the trays, and assemble. Test that the trays move correctly.

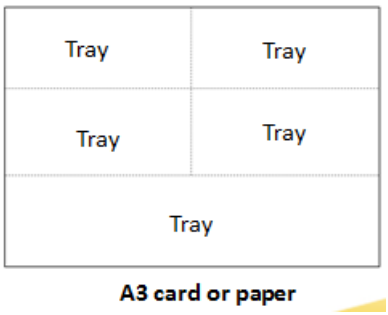
Expected Student Response

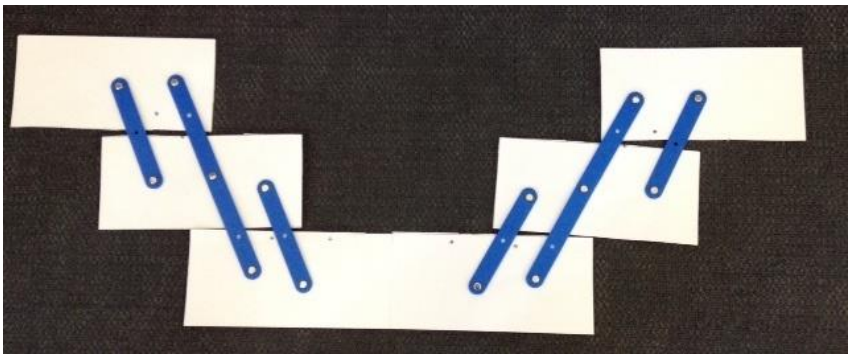
- The bars connecting the trays are parallel.

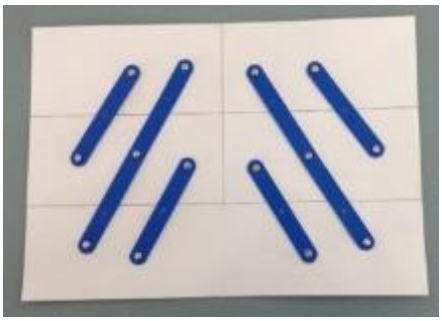
- On each tray, the rivets are horizontally aligned.
- When the box is fully open or fully closed, there is also vertical alignment of the rivets (pivot points).

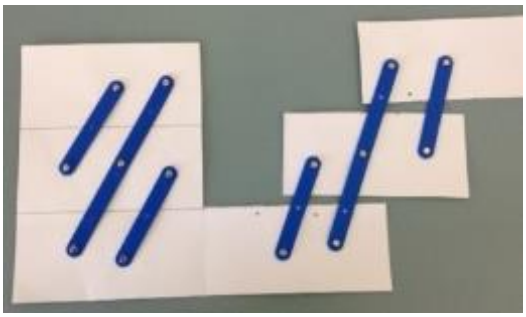
Teacher Notes

In constructing their models, students will especially need to consider both the vertical and horizontal alignment of the paper fasteners. The students whose models are shown below had only partly appreciated this.







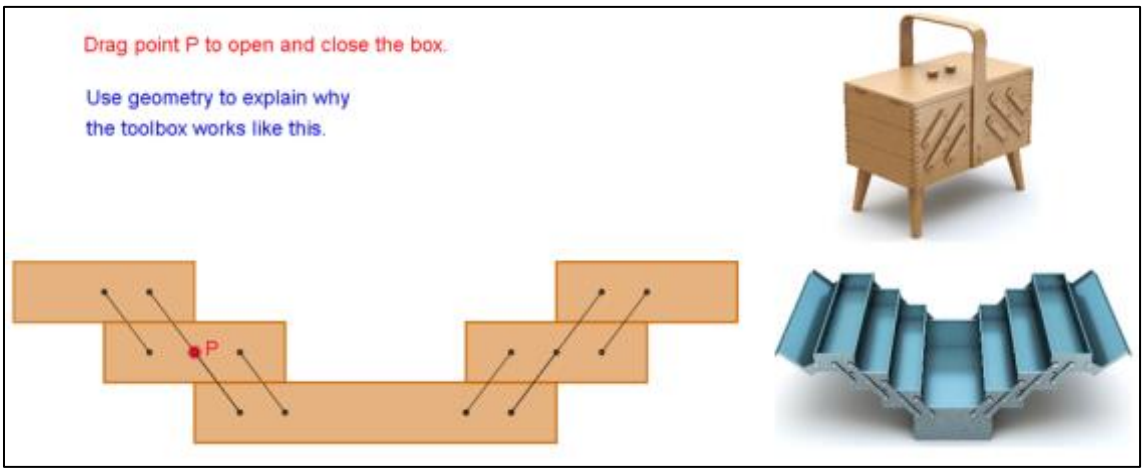


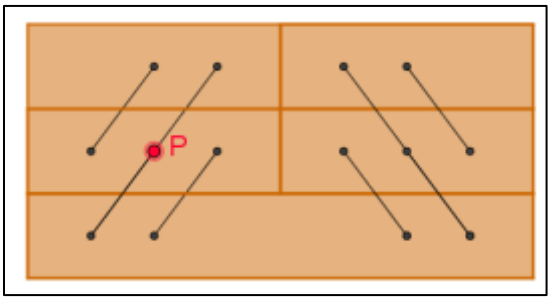
Using a computer simulation

The GeoGebra file file *ST1_Yr8_4b_Toolbox.ggb* (or the GeoGebra Tube link https://ggbm.at/V4eMwGSk) simulates the opening and closing of the box. Focus students' attention on the position of the rivets that attach the brackets to the box (shown in the simulation as black dots), and how the trays always stay horizontal as they move.

Expected student response

- The rows of rivets stay horizontal as the box is opened and closed.
- The horizontal distances between the two rivets on each tray are all the same.
- The middle bar is twice as long as the side bars, and the central rivet is at the centre.
- When the box is fully opened or fully closed, the rivets at the ends of the brackets stay in a vertical line.
- The bars on each side stay parallel.





Why Do the Toolbox Trays Stay Horizontal?

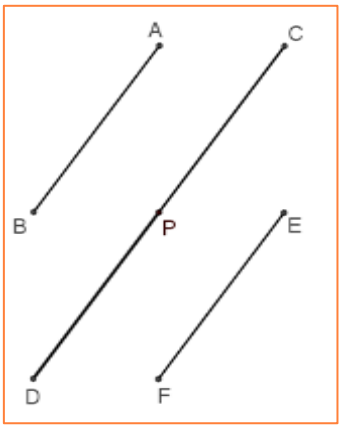
Gather ideas from students about why the trays stay horizontal. It will help to draw in the horizontal lines as shown below, so that students see the parallelograms.

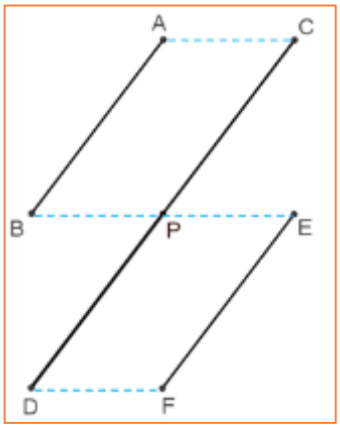
Students can argue visually at this stage: because all the shapes are parallelograms, the segments AC, BP, PE and DF are all parallel. DF is on the bottom tray and so is kept horizontal.

In brief, the trays stay horizontal because:

- the linkage is constructed of quadrilaterals with opposite sides of equal length
- quadrilaterals with sides of equal length are parallelograms (this is the main topic below)
- the bottom tray is horizontal, and the other two trays are parallel to it.

Ask students how they know that the quadrilaterals are parallelograms. This motivates the next lesson segment.





Teacher Notes

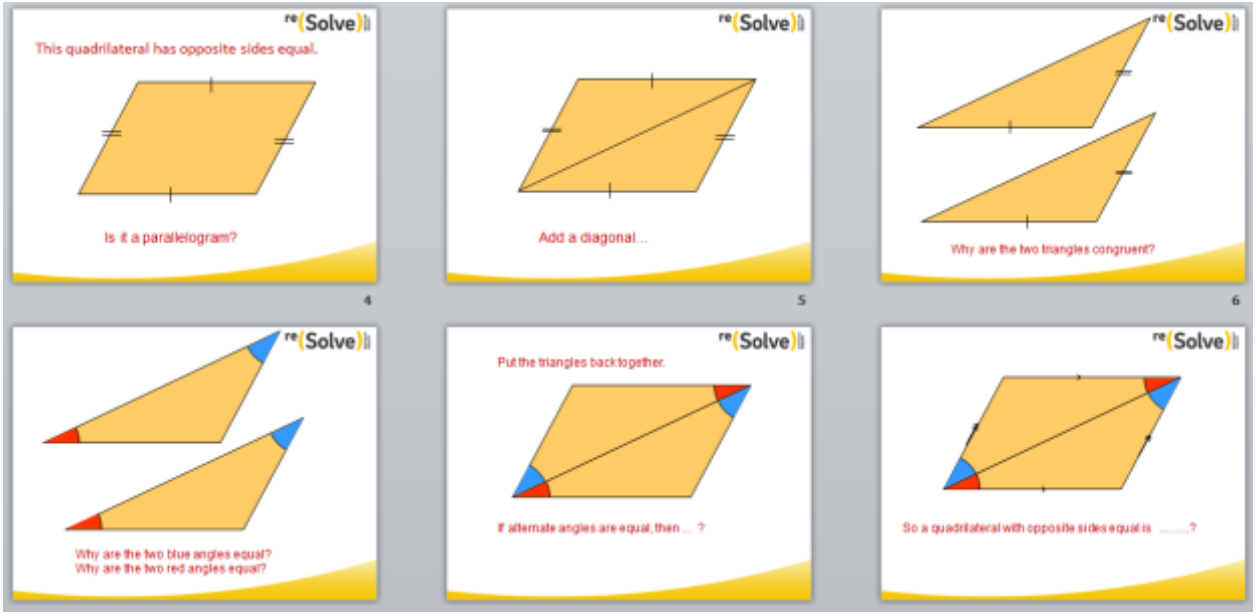
In more detail, the line segments AB, CD and EF represent the brackets attached to the trays. The box is designed so that AB = CP = PD = EF. Students can draw line segments AC, BE and DF. Their observations from making their own physical model and from using the computer simulation will have enabled them to see that the box is also designed so that AC = BP = PE = DF. Quadrilaterals ABPC and PDFE therefore have opposite sides equal (in every position). Hence they are parallelograms.

DF is always horizontal when the tool box is being used (placed on a level surface), This means that PE is horizontal, which means that BP (part of the same bar) is horizontal, which means that AC is horizontal. So all the trays are horizontal when the bottom one is.

Showing the quadrilaterals are parallelograms

- Ask students to open their individual or shared copies of the slide show *ST1_Yr8_3c_IsItParallelogram.ppt*.
- Students will recall that the definition of a parallelogram is a quadrilateral with opposite sides parallel. Starting with this definition, the slideshow leads students through visual reasoning that a quadrilateral with opposite sides equal is a parallelogram.
- The reasoning requires use of congruent triangles as in the Australian Curriculum: Mathematics (Year 8).

- Visual reasoning through the slideshow will be adequate for most students. More advanced students could write out their reasoning using the labelled diagram above as on Teacher Sheet 1 – Parallelograms and Deductive Reasoning.



Expected student responses

- The two triangles are congruent, because they have three sides of the same length.
- Hence, the matching angles are equal.
- The equal angles form pairs of alternate angles.
- Hence, the opposite sides are parallel.

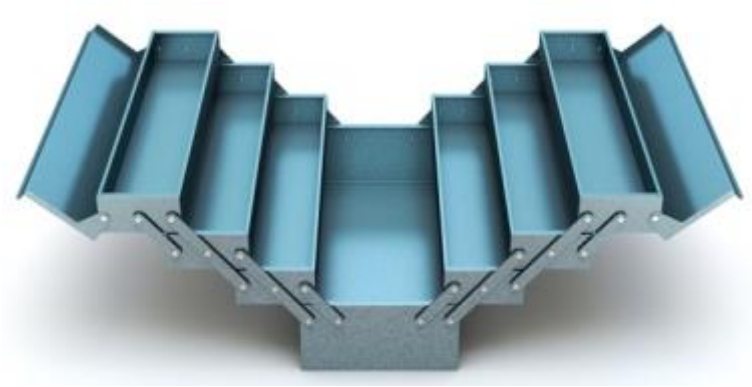
Conclusion

- If both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
- The arrangement of the brackets and rivets makes quadrilaterals with opposite sides equal. So these form parallelograms. This ensures that the trays stay horizontal as the toolbox is opened.

Toolbox

Name: _____

Toolboxes as shown below are called cantilever toolboxes because of the way the trays open out. The wooden box on the left has two open-out trays on each side, but the metal box has three trays on each side.



Making a physical model

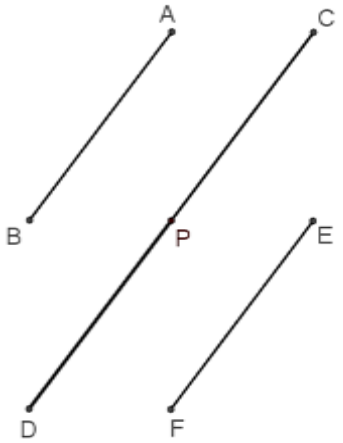
- Construct a 2-dimensional model of a toolbox that has two trays on each side, like the wooden box
- Use rectangular pieces of card to represent the trays, strips to represent the bars and paper fasteners to connect the bars to the trays.
- In constructing your model, you will need to consider both the vertical and horizontal alignment of the paper fasteners.
- Does your model work in the expected way? If not, what should be changed?

Using a computer simulation

- Open the GeoGebra file *Toolbox*. This shows a simulation of the opening and closing of a toolbox.
- Drag point P to open and close the toolbox.
- Carefully watch the positions of the black dots that represent the rivets that attach the brackets to the trays.

Explaining why the toolbox works the way it does

- Using the diagram below (adding any extra line segments you think are useful) explain how the geometry ensures that the toolbox opens and closes the way it does.



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Teacher Sheet 1 - Parallelograms and Deductive Reasoning

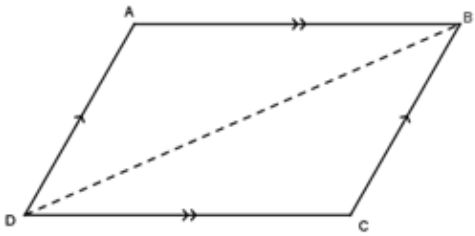
A parallelogram is defined as a quadrilateral with opposite sides parallel. Shown below is a proof that opposite sides are also equal.

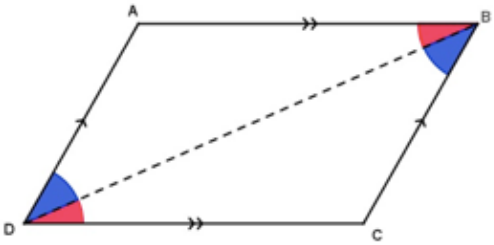
The converse statement is that a quadrilateral with opposite sides equal is a parallelogram (i.e. has opposite sides parallel). This is also proved below.

More advanced students could be introduced to the concept of the converse of a theorem. The converse of a true statement is NOT always true. For example, a quadrilateral with opposite sides parallel has diagonals of equal length but a quadrilateral with diagonals of equal length might not have opposite sides parallel.

Showing that both pairs of opposite sides of a parallelogram are equal in length

$ABCD$ is a parallelogram. Hence $AB \parallel DC$ and $AD \parallel BC$. Show that $AB = DC$ and $AD = BC$.

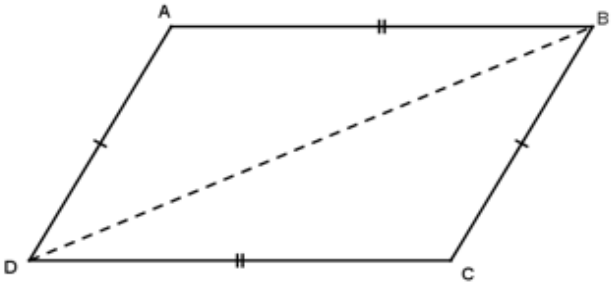


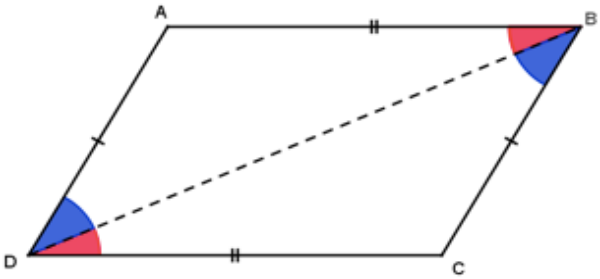


Proof: Draw the diagonal BD
In ΔABD and ΔCDB,
BD is common
$\angle ABD = \angle CDB$ (alternate angles)
$\angle ADB = \angle CBD$ (alternate angles)
So $\Delta ABD \equiv \Delta CDB$ (A.S.A.)
So $AB = DC$ and $AD = BC$

Showing that if both pairs of opposite sides of a quadrilateral are equal in length, then it is a parallelogram.

$ABCD$ is a quadrilateral. $AB = DC$ and $AD = BC$. Show that $ABCD$ is a parallelogram.





Draw the diagonal BD
In ΔABD and ΔCDB,
BD is common
$AB = DC$ and $AD = BC$
So $\Delta ABD \equiv \Delta CDB$ (S.S.S.)
So $\angle ABD = \angle CDB$ (matching angles in congruent triangles)
So AB is parallel to DC (alternate angles are equal)
Similarly $\angle ADB = \angle CBD$ (matching angles in congruent triangles)
So AD is parallel to BC (alternate angles are equal)
Hence $ABCD$ has opposite sides parallel and so it is a parallelogram.

This proves that a quadrilateral with both pairs of opposite sides equal is a parallelogram.

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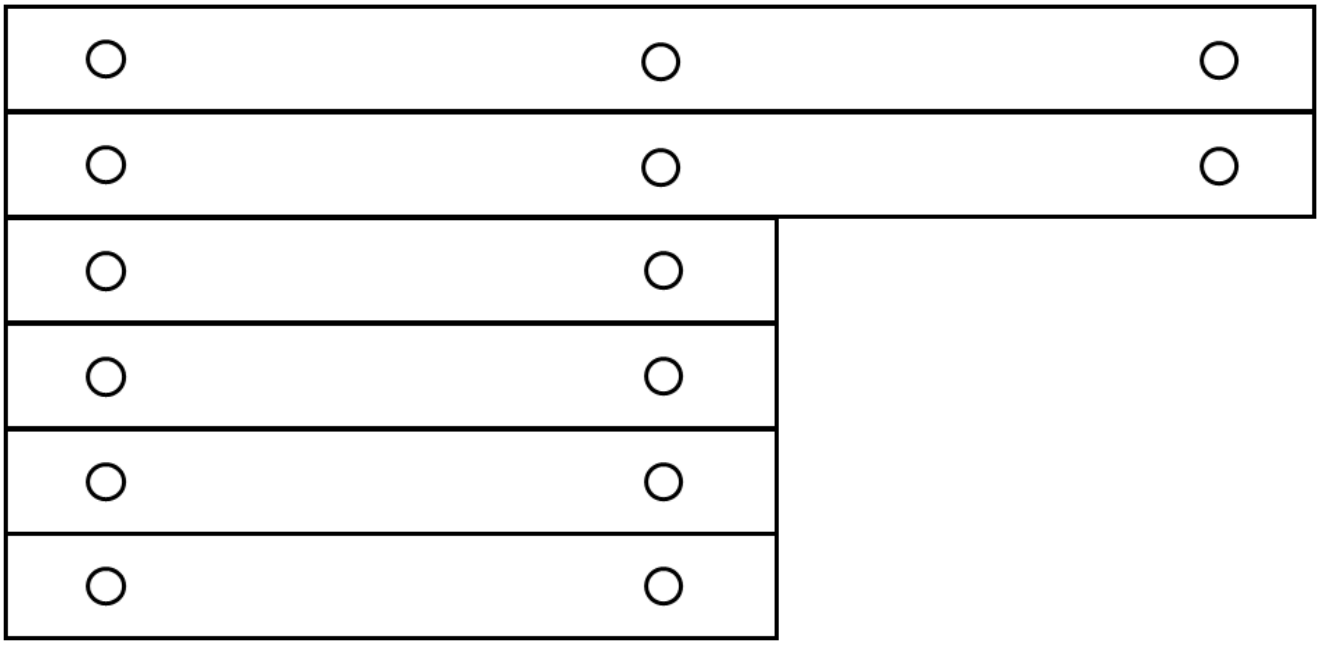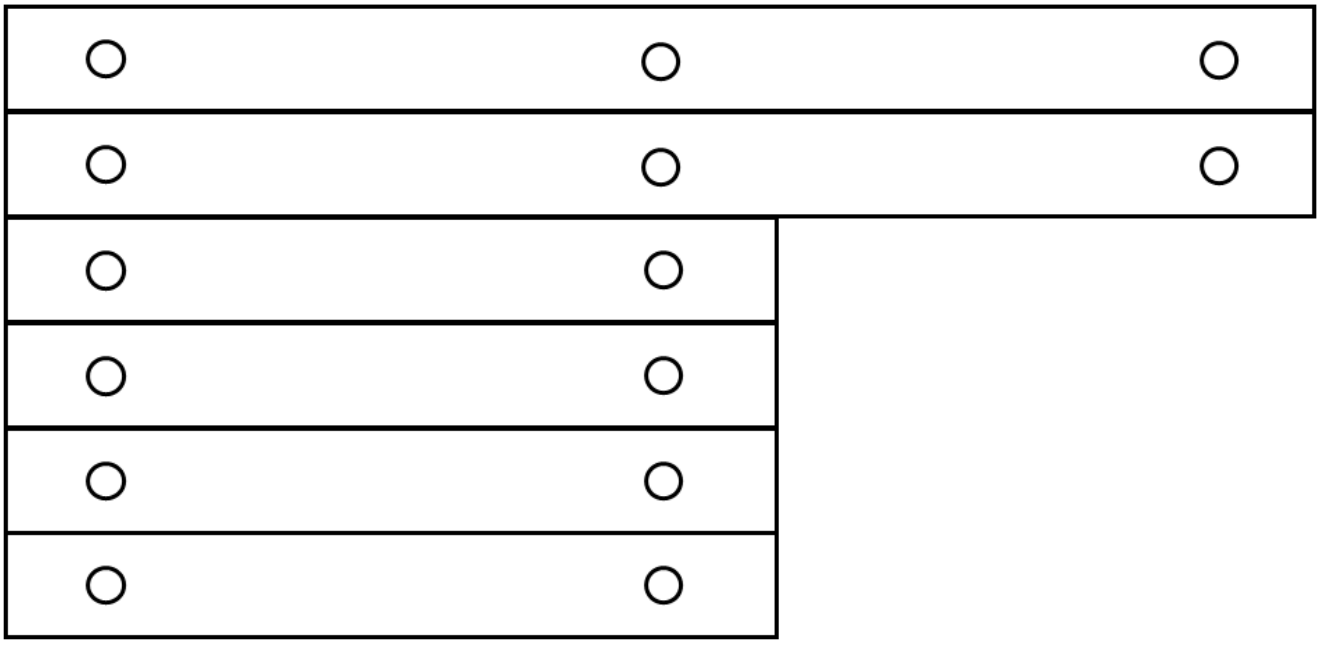
Template for Toolbox Brackets (two sets)