INTRODUCTION

re(Solve) MATHS BY INQUIRY

Prisms and Pyramids

Y7

Summary of learning goals

- This sequence provides an accessible context for students to use simple algebra. Students build their skills in algebra by developing algebraic rules for the numbers of faces, edges and vertices in prisms and pyramids. They make deductions about unknown prisms and pyramids from these rules (which links to equation solving) and then use their algebraic expressions to show that Euler's formula works for all prisms and pyramids. Students also build their spatial skills through construction of pyramids and prisms and learn about a new class of polyhedra: antiprisms.

Australian Curriculum: Mathematics (Year 7)

ACMNA175: Introduce the concept of variables as a way of representing numbers using letters.

ACMNA176: Create algebraic expressions and evaluate them by substituting a given value for each variable.

ACMNA177: Extend and apply the laws and properties of arithmetic to algebraic terms and expressions.

ACMMG161: Draw different views of prisms and solids formed from combinations of prisms.

Summary of lessons

Who is this sequence for?

- These lessons are designed for students who are beginning to learn algebra. Students will need to understand that pronumerals stand for numbers and know the most basic conventions of algebra. The emphasis is on developing algebraic relationships through visualisation, rather than through looking at patterns in tables. For the Euler's formula task, students must also be able to collect like terms. For example, they will need to be confident that $2b \neq b + 2$ and be able to simplify expressions such as $(b + 2) - 2b$. The lesson can involve solving very simple equations.

Lesson 1: Faces, Edges and Vertices

Students determine the numbers of faces, edges and vertices of prisms and pyramids, in terms of the number of sides of the base shape. They use these results to show that Euler's formula holds for all prisms and pyramids.

Lesson 2: Antiprisms

Students learn about a new class of polyhedra called antiprisms. They build on Lesson 1 to determine the numbers of faces, edges and vertices of an antiprism when given the number of sides in the base shape.

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Australian Academy of Science

Reflection on this sequence

Rationale

A focus on the properties of familiar three-dimensional (3D) objects has been selected as the vehicle for describing relationships with algebra. Students visualise and work with concrete objects, algebraically expressing relationships between the number of sides in the base shape and the numbers of faces, edges and vertices of the 3D object. This provides meaning and purpose for the algebra. Application of students' algebraic rules to Euler's formula could be a first opportunity for them to prove a result using algebra.

reSolve mathematics is purposeful

- These lessons emphasise the links between algebra and the physical world; in this case, 3D objects. Students develop skills in reasoning and communication as they make and justify generalisations, verbally and algebraically. The geometric situation illustrates the meaning of the algebra, and the algebra can describe the geometry and make predictions about it. Students develop a sense of why algebra is useful.

reSolve tasks are inclusive and challenging

- Using constructed prisms and pyramids as a vehicle for developing algebraic understanding provides students with an accessible entry point to generalisation. There is an opportunity to link formal equation-solving processes with intuitive methods.
- Students are challenged to manipulate algebraic variables to find unknowns, and to produce two short algebraic proofs.

reSolve classrooms have a knowledge-building culture

- The sequence is a guided investigation during which students can work together to discover patterns and make deductions. Each task builds on the previous task to help students move towards a greater understanding of what algebra can do.

Acknowledgements

All images of antiprisms used in Lesson 2 are reproduced from the Florida Center for Instructional Technology *ClipArt ETC* Collection; see https://etc.usf.edu/clipart/.

LESSON 1

reSolve: MATHS BY INQUIRY

Prisms and Pyramids

Faces, Edges and Vertices

Y7

About this lesson

Students determine the numbers of faces, edges and vertices of prisms and pyramids, in terms of the number of sides of the base shape. They use these results to show that Euler's formula holds for all prisms and pyramids.

Australian Curriculum: Mathematics (Year 7)

ACMNA175: Introduce the concept of variables as a way of representing numbers using letters.

ACMNA176: Create algebraic expressions and evaluate them by substituting a given value for each variable.

ACMNA177: Extend and apply the laws and properties of arithmetic to algebraic terms and expressions.

ACMMG161: Draw different views of prisms and solids formed from combinations of prisms.

Mathematical purpose

- Students use simple algebra in a meaningful context. They express algebraically the numbers of faces, vertices and edges in polygonal prisms and pyramids. They then use these algebraic expressions in an equation-solving mode to identify mystery shapes. They also use algebra in a generalisation mode to prove that Euler's formula is true for all prisms and pyramids. Students build their spatial visualisation skills through consideration of arbitrary pyramids and prisms.

Learning intention

- To investigate how algebra can be used to describe relationships, make predictions and show that relationships are always true.

Time

Two lessons of approximately 1 hour each.

Vocabulary

- edges
- Euler's formula
- faces
- polyhedra/polyhedron
- prism
- pyramid
- vertex/vertices

Resources

- reSolve PowerPoint *1a Faces, Edges and Vertices* (for display)
- one class set of Geoshapes (optional)
- toothpicks or straws and Blu-Tack (optional)
- nets of hexagonal prism and hexagonal pyramid (optional)
- Student Sheet 1 – Prisms and Pyramids
- Student Sheet 2 – Mystery Shapes

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Teacher background information

Students are often asked to construct tables and look for patterns to determine algebraic relationships. This can be a very useful strategy; however, it may hide the underlying mathematical structure. Hence, this resource is designed to focus on mathematical structure rather than spotting patterns. The structure that students notice is then expressed algebraically.

Making 3D shapes

Making models of shapes helps students notice their different properties:

- Building shapes using Geoshapes or Polydrons focuses attention on the number and shape of faces.
- Drawing or modelling using straws or toothpicks focuses attention on the number of edges and vertices.

Noticing things about prisms

Resources: Show students slide 2 of reSolve PowerPoint *1a Faces, Edges and Vertices*.

The open questions *What do you notice?* and *Why?* allow for a variety of responses. The numbers of faces, edges and vertices are given to help focus students.

Possible student responses:

- The base shape (a pentagon) has five sides.
- The pentagonal prism has 10 vertices because the pentagon is repeated on the top and bottom.
- The pentagonal prism has seven faces because there is a pentagon on the top and bottom, joined by a rectangle on each of the five sides.
- The pentagonal prism has 15 edges because each pentagon has five sides, and each vertex of the pentagon is joined with an edge to the corresponding vertex of the other pentagon.
- The number of faces + number of vertices = number of edges + 2 (Note: this is Euler's formula, which will be discussed later in the lesson. If students do not notice this now, it can be left until then.)

Show students slide 3. Encourage students to first visualise a hexagonal prism to determine the numbers of faces, edges and vertices.

Show students slide 4. The phrase 'lots of sides' focuses attention on the relationships between the number of sides of the base shape and the numbers of faces, edges and vertices, without rushing to an algebraic formulation.

Possible student responses:

- The number of vertices of a prism is always double the number of sides of the base shape.
- The number of faces of a prism is always two more than the number of sides of the base shape.
- The number of edges of a prism is always three times the number of sides of the base shape.

Noticing things about pyramids

Show students slide 5.

Possible student responses:

- The pentagonal pyramid has six vertices because there is one vertex above the base shape.
- The pentagonal pyramid has six faces because there is a pentagon on the bottom and each side of the pentagon is joined to the vertex of the pyramid by a triangle.
- The pentagonal pyramid has 10 edges because the pentagon has five sides, and each vertex of the pentagon is joined with an edge to the vertex of the pyramid.

Show students slide 6. Encourage students to visualise a hexagonal pyramid to determine the numbers of faces, edges and vertices.

Show students slide 7.

Possible student responses:

- The number of vertices of a pyramid is always one more than the number of sides of the base shape.
- The number of faces of a pyramid is always one more than the number of sides of the base shape.
- The number of edges of a pyramid is always double the number of sides of the base shape.

Formalising

Resources: Provide students with Student Sheet 1 – Prisms and Pyramids.

After students have discussed and agreed on their informal observations about the numbers of faces, edges and vertices of prisms and pyramids, ask them to consider a prism and a pyramid whose base has *b* sides.

Possible student response:

Property	Rule in words	Why it works	Rule in algebra
Number of faces of prism	Two more than the number of base sides	There is one face for each base side, then a top face and a bottom face.	$f = b + 2$
Number of edges of prism	Three times the number of base sides	The top and bottom faces each have *b* edges, and there is one vertical edge for each base side.	$e = 3b$ or $e = b \times 3$ or $e = b + b + b$
Number of vertices of prism	Two times the number of base sides	There is one vertex for each side of the top face and one for each side of the bottom face.	$v = 2b$ or $v = b \times 2$ or $v = b + b$
Number of faces of pyramid	One more than the number of base sides	There is one face for each base side, then a bottom face.	$f = b + 1$
Number of edges of pyramid	Two times the number of base sides	The bottom face has *b* edges, and there is one slanted edge for each base side.	$e = 2b$ or $e = b \times 2$ or $e = b + b$
Number of vertices of pyramid	One more than the number of base sides	There is one vertex for each side of the bottom face plus one extra vertex at the top.	$v = b + 1$

Applying

Resources: Student Sheet 2 – Mystery Shapes provides a chance for students to use the rules that they have just discovered.

Some students may prefer to work with their rules in words rather than in algebra. Encourage linking algebra and the physical reality that the variables represent. This task can provide an opportunity for simple equation solving. The first four rows require relatively simple substitution. The last seven rows will require some extra reasoning or trial and error to work out whether the 3D shape is a pyramid or a prism (or impossible).

Possible student response:

Prism or pyramid?	Number of sides of base (b)	Number of faces (f)	Number of edges (e)	Number of vertices (v)	How did you work this out?
prism	20	22	60	40	Substitute $b = 20$ into each formula.
prism	18	20	54	36	First, subtract 2 from the number of faces to work out b, then substitute.
pyramid	16	17	32	17	First, subtract 1 from the number of vertices to work out b, then substitute.
pyramid	11	12	22	12	First, divide the number of edges by 2 to work out b, then substitute.
prism			22		Impossible: 22 is not a multiple of 3 and the number of edges of a prism is of the form $3b$.
prism	9	11	27	18	Since the number of edges is a multiple of 3 but not a multiple of 2, the shape is a prism.
pyramid	10	11	20	11	The number of faces is the same as the number of vertices, so this must be a pyramid.
prism	50	52	150	100	The number of vertices is twice the number of base sides, so this is a prism.
pyramid	50	51	100	51	The number of faces is one more than the number of base sides, so this is a pyramid.
prism	8	10	24	16	The number of edges is a multiple of both 2 and 3, so students will need to refer to the other columns. For a prism, $3b = 24$, giving $b = 8$ and $f = 10$. For a pyramid, $2b = 24$, giving $b = 12$ and $v = 13$.
pyramid	12	13	24	13	

Another relationship: Euler's Rule

If students did not notice the relationship *number of faces* + *number of vertices* = *number of edges* + 2 in the first part of the lesson, encourage them to look at a shape of their choice and find a relationship between the numbers of faces, edges and vertices. The relationship may be expressed in different ways, but all should notice the same result.

Teacher notes:

- This is Euler's rule. It applies to all polyhedra that have no 'holes' going through them. There is a proof, written for teachers or advanced students, at https://plus.maths.org/content/eulers-polyhedron-formula.
- Using the results we have obtained for prisms and pyramids, we can prove Euler's rule in these special cases. This is likely to be the first time students have substituted one algebraic expression for another and could be a first opportunity for students to prove a result using algebra.
- We suggest that the proof for prisms be explicitly demonstrated to students, and that they then be asked to write down a similar proof for pyramids.

Using f, e and v for the numbers of faces, edges and vertices, respectively, ask students to write Euler's rule algebraically. Some possible variations are:

- $f + v - e = 2$
- $f + v = e + 2$
- $e = f + v - 2$

Proof for prisms

We know that, for prisms:

$f = b + 2$;

$v = 2b$; and

$e = 3b$.

Hence:

$$\begin{aligned} f + v - e &= (b + 2) + (2b) - (3b) \\ &= b + 2b - 3b + 2 \\ &= 2 \end{aligned}$$

Ask students to write down a similar proof for pyramids.

Possible student response:

We know that, for pyramids:

$f = b + 1$;

$v = b + 1$; and

$e = 2b$.

Hence:

$$\begin{aligned} f + v - e &= (b + 1) + (b + 1) - (2b) \\ &= b + b - 2b + 1 + 1 \\ &= 2 \end{aligned}$$

Prisms and Pyramids

Name: ______________________________

In the table below:

- Describe in words how you could work out the numbers of faces, edges and vertices of a prism if you know the number of sides of the base shape. Explain why this works.
- Describe in words how you could work out the numbers of faces, edges and vertices of a prism if you know the number of sides of the base shape. Explain why this works.
- Write these rules algebraically. Assume that the base shape has *b* sides.

Property	Rule in words	Why it works	Rule in algebra
Number of faces of prism			$f =$
Number of edges of prism			$e =$
Number of vertices of prism			$v =$
Number of faces of pyramid			$f =$
Number of edges of pyramid			$e =$
Number of vertices of pyramid			$v =$

Mystery Shapes

Name: ____________________________________

Each of the three-dimensional shapes in the table below is either a prism, a pyramid or, in one case, impossible. Use the rules you found previously for the numbers of faces, edges and vertices to fill in the blanks. You will need to work out whether the last six shapes are prisms or pyramids.

You may need extra working out space. Explain in the last column how you worked out each result.

Prism or pyramid?	Number of sides of base (b)	Number of faces (f)	Number of edges (e)	Number of vertices (v)	How did you work this out?
prism	20				
prism		20			
pyramid				17	
pyramid			22		
prism			22		
			27		
		11		11	
	50			100	
	50	51			
		10	24		
			24	13	

LESSON 2

re(Solve) MATHS BY INQUIRY **Prisms and Pyramids**

Antiprisms

Y7

About this lesson

Students learn about a new class of polyhedra called antiprisms. They build on Lesson 1 to determine the numbers of faces, edges and vertices of an antiprism when given the number of sides in the base shape.

Australian Curriculum: Mathematics (Year 7)

ACMNA175: Introduce the concept of variables as a way of representing numbers using letters.

ACMNA176: Create algebraic expressions and evaluate them by substituting a given value for each variable.

ACMNA177: Extend and apply the laws and properties of arithmetic to algebraic terms and expressions.

Mathematical purpose

- As in Lesson 1, students use algebra to express relationships in three-dimensional geometry. They build their spatial visualisation skills and learn about antiprisms.

Learning intention

- To investigate the many types of polyhedra.
- To investigate how algebra can be used to describe relationships, make predictions and show that relationships are always true.

Time

A lesson of approximately 1 hour.

Vocabulary

- antiprism

Resources

- reSolve PowerPoint *2a Antiprisms* (for display)
- Student Sheet 1 – More Mystery Shapes
- Geoshapes or reSolve PDF *2b Antiprism Net* (printed on A3 paper)

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Teacher background information

Antiprisms are related to prisms in that two parallel polygons are joined by other polygons. Unlike prisms, they do not have the same cross-section at all positions between the polygons. If the polygons joining the two parallel polygons are equilateral triangles, then the antiprism satisfies the same conditions as the 13 Archimedean solids, in that all faces are regular polygons and all vertices are identical.

Noticing things about antiprisms

Making an antiprism

Resources: Provide students with reSolve PDF *2b Antiprism Net*.

This contains two hexagons and a strip of 15 equilateral triangles. Alternatively, provide students with two hexagons from a set of Geoshapes and at least 15 equilateral triangles joined as a long strip.

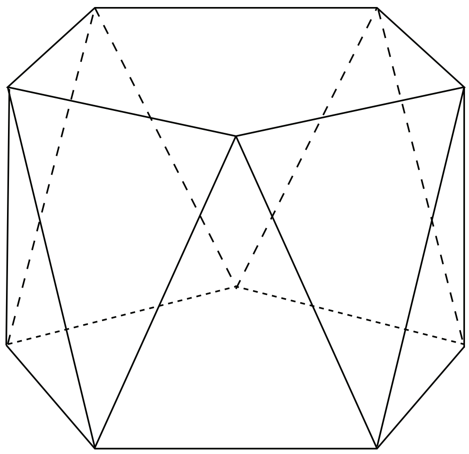
Construct a polyhedron so that the two hexagons are parallel to each other and use as many of the triangles in the strip as is necessary to join the two hexagons.

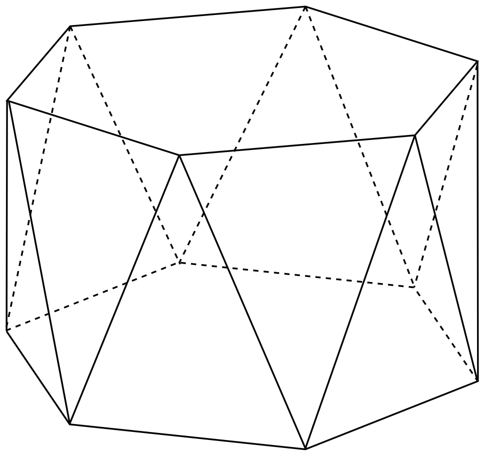
Teacher note:

- Drums are often antiprisms because the triangular faces are rigid.

Pose the questions: *How do you need to arrange the two hexagons in order to make a polyhedron?*
How many triangles do you need to join the hexagons?

Show students slides 2 and 3 of reSolve PowerPoint *2a Antiprisms*, which show pictures of pentagonal and hexagonal antiprisms. As in Lesson 1, encourage students to explain how they can determine the numbers of faces, vertices and edges.





Possible student responses:

- The hexagonal antiprism has 12 vertices because the hexagon is repeated on the top and the bottom.
- The hexagonal antiprism has 14 faces because there is a hexagon on the top and bottom, joined by a strip of 12 triangles.
- The hexagonal antiprism has 24 edges because each hexagon has six sides, and each vertex of one hexagon is joined by two edges to a vertex of the other hexagon.

Show students slide 4. The phrase 'lots of sides' focuses attention on the relationships between the number of sides of the base shape and the numbers of faces, edges and vertices, without rushing to an algebraic formulation.

Possible student responses:

- The number of vertices of an antiprism is always double the number of sides of the base shape.
- The number of faces of an antiprism is always two more than double the number of sides of the base shape.
- The number of edges of an antiprism is always four times the number of sides of the base shape.

Formalising the features of antiprisms

After students have discussed and agreed on their informal observations about the numbers of faces, edges and vertices of an antiprism, ask them to consider an antiprism whose base has b sides. As in Lesson 1, develop algebraic rules for the numbers of faces, edges and vertices when given the number of sides on the base shape. Also ask students to verify Euler's formula for antiprisms; that is, $f + v - e = 2$.

Possible student responses:

- The number of vertices is given by $v = 2b$.
- The number of faces is given by $f = 2b + 2$.
- The number of edges is given by $e = 4b$.
- Proof of Euler's formula for antiprisms

  We know that, for prisms:

  $f = 2b + 2$;

  $v = 2b$; and

  $e = 4b$.

  Hence:

  $$\begin{aligned} f + v - e &= (2b + 2) + (2b) - (4b) \\ &= 2b + 2b - 4b + 2 \\ &= 2 \end{aligned}$$

Applying

Resources: Student Sheet 1 – More Mystery Shapes provides a chance for students to use the rules that they have just discovered.

Some students may prefer to work with their rules in words rather than in algebra. Encourage linking algebra and the physical reality that the variables represent. This task can provide an opportunity for simple equation solving. The first three rows require relatively simple substitution. The last three rows will require some extra reasoning or trial and error to work out whether the 3D shape is a prism, a pyramid or an antiprism.

<table>
<tr><th>Prism, antiprism or pyramid?</th><th>Number of sides of base (b)</th><th>Number of faces (f)</th><th>Number of edges (e)</th><th>Number of vertices (v)</th><th>How did you work this out?</th></tr>
<tr><td>antiprism</td><td>20</td><td>42</td><td>80</td><td>40</td><td>Substitute $b = 20$ into each formula.</td></tr>
<tr><td>antiprism</td><td>9</td><td>20</td><td>36</td><td>18</td><td>First, subtract 2 from the number of faces and divide by 2 to work out b, then substitute.</td></tr>
<tr><td>antiprism</td><td>4</td><td>10</td><td>16</td><td>8</td><td>First, divide the number of edges by 2 to work out b, then substitute.</td></tr>
<tr><td>pyramid</td><td>15</td><td>16</td><td>30</td><td>16</td><td rowspan="3">In each case, the two pieces of information must be combined to determine the shape. A shape that has 16 faces could be:
• An antiprism whose base has 7 sides;
• A prism whose base has 14 sides; or
• A pyramid whose base has 15 sides.
Substituting these into the formulas for each shape gives the solution.</td></tr>
<tr><td>prism</td><td>14</td><td>16</td><td>42</td><td>28</td></tr>
<tr><td>antiprism</td><td>7</td><td>16</td><td>28</td><td>14</td></tr>
</table>

More Mystery Shapes

Name: ____________________________________

Each of the shapes in the table below is either a prism, a pyramid or an antiprism. Use the rules you found previously for the numbers of faces, edges and vertices to fill in the blanks. For the last three rows, you will need to work out what is the shape.

You may need extra working out space. Explain in the last column how you worked out each result.

Prism, antiprism or pyramid?	Number of sides of base (b)	Number of faces (f)	Number of edges (e)	Number of vertices (v)	How did you work this out?
antiprism	20				
antiprism		20			
antiprism			16		
		16	30		
		16		28	
		16		14	