re(Solve) MATHS BY INQUIRY Tens and Units

Summary of learning goals

- Students come to appreciate the power of algebra for generalising results from arithmetic. They connect arithmetic operations with algebraic notation and visualisations. Each lesson commences with an observation made using arithmetic that students then justify and extend using algebra.

Australian Curriculum: Mathematics (Year 8)

ACMNA190: Extend and apply the distributive law to the expansion of algebraic expressions.

ACMNA191: Factorise algebraic expressions by identifying numerical factors.

ACMNA192: Simplify algebraic expressions involving the four operations.

ACMSP206: Explore the practicalities and implications of obtaining data through sampling using a variety of investigative processes.

Summary of lessons

Who is this sequence for?

- This set of lessons is designed to consolidate skills in algebra, including collecting like terms and expanding and factorising using the distributive law. The resources emphasise the importance of algebra for generalising and justifying arithmetic results. It is assumed that students have some familiarity with algebraic notation.

Lesson 1: Mathematical Mind Reading

This lesson uses algebra to explain results in arithmetic by expressing two- and three-digit numbers in the general form $10a + b$ or $100a + 10b + c$. Students examine a well-known ‘trick’ involving reversing digits and explain it using algebra. They extend the activity to three-digit numbers and sustain their learning using algebra to predict what will happen in a related activity.

Lesson 2: Reverse and Add

This lesson extends from the first lesson in the sequence. Students use algebra to explain results in arithmetic by expressing two- and three-digit numbers in the general form $10a + b$ or $100a + 10b + c$. Students start by reversing the digits of a two-digit number, adding it to the original number and observing that the result is always a multiple of 11. Links are made to ideas in statistics by randomly generating a large sample of two-digit numbers.

reSolve: Mathematics by Inquiry is funded by the Australian Government and is managed by the Australian Academy of Science.

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Reflection on this sequence

Rationale

Approaching algebra as generalised arithmetic shows students the power of algebra for abstracting number. This focus on algebra as generalised arithmetic is typically under-represented in secondary mathematics in favour of more time spent on functions and equations. The lessons build on place value to develop generalised expressions for multidigit numbers, a foundational element for understanding number theory.

reSolve mathematics is purposeful

- This sequence supports a rich interpretation and enactment of the Australian Curriculum: Mathematics, providing fun and engaging ways to understand the algebraic content of the Curriculum. The lessons explore mathematics as a creative and imaginative endeavour, emphasising the entertaining applications of mathematics and the understanding of the mathematical foundation beneath common ‘tricks’ that they might have encountered before.

reSolve tasks are inclusive and challenging

- The tasks in this sequence activate existing knowledge, develop new knowledge and explore relationships between key ideas in the Australian Curriculum. The lessons apply algebraic concepts to ideas of place value learned in the primary years, which allows for a low floor (engaging with algebraic notation) and a high ceiling (generalisation of algebraic concepts). Students are then required to navigate different statistical and mathematical software in ways that they are unlikely to be experienced with. The lessons include a wide variation of prompts and programs to suit students.

reSolve classrooms have a knowledge-building culture

- Each task in this sequence begins by inspiring curiosity and intrigue through a shared classroom experience that promotes higher-order thinking through the role of both teacher and student. Students build understanding through collaborative inquiry, action and reflection. The sequence encourages students to challenge their existing conceptions and to use their mistakes as a vehicle for further learning.

LESSON 1

reSolve: Maths by Inquiry — Tens and Units

Mathematical Mind Reading

Y8

About this lesson

This is one of two lessons that use algebra to explain results in arithmetic by expressing two- and three-digit numbers in the general form $10a + b$ or $100a + 10b + c$. Students examine a well-known 'trick' involving reversing digits and explain it using algebra. They extend the activity to three-digit numbers and sustain their learning using algebra to predict what will happen in a related activity.

Australian Curriculum: Mathematics (Year 8)

ACMNA190: Extend and apply the distributive law to the expansion of algebraic expressions.

ACMNA191: Factorise algebraic expressions by identifying numerical factors.

ACMNA192: Simplify algebraic expressions involving the four operations.

Mathematical purpose

- The resource develops students' understanding of algebra as generalised arithmetic. Students learn to express two- and three-digit numbers in the general form $10a + b$ or $100a + 10b + c$ and use this to explain the results of arithmetic operations involving numbers with their digits reversed. The task links the ideas of place value learned in the primary years with algebraic reasoning.

Learning intention

- To understand and explain algebraically how a mathematical trick works.

Time

Two lessons of approximately 1 hour each.

Vocabulary

- coefficient
- variable

Resources

- reSolve PowerPoint *1a Mathematical Mind Reader*
- reSolve PowerPoint *1b How Many Blocks*?

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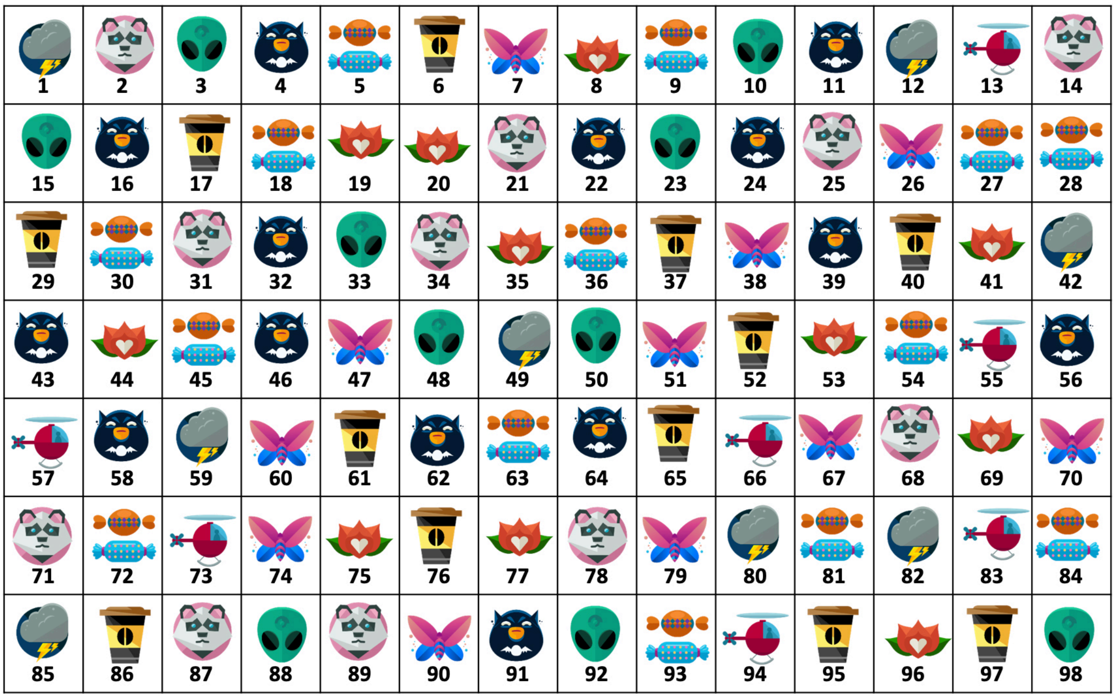
The reSolve mathematical mind reader

Resources: Show the reSolve PowerPoint *1a Mathematical Mind Reader*.

Slide 1 instructs students to write down a two-digit number, then reverse the digits and subtract the smaller number from the larger number. Slide 2 shows 98 numbers, each with an icon associated with it. Students are asked to remember the icon of their number. On slide 3, their icon ‘magically’ appears.



The activity is then repeated with a rearranged set of icons.

Ask students to discuss how this trick might work and share some strategies.

Investigating the answers

Collect some responses from students, asking them to state their original number, its reverse, and the result of the subtraction. It may be helpful to record the responses in a table, as shown.

Number	Reverse	Result of subtraction
43	34	9
61	16	45
13	31	18
97	79	18
38	83	45

Look for some patterns in the table and generalise.

Every answer is a multiple of 9. The answer is nine times the difference between the two digits.

Explaining the 'magic'

Discuss with students how a two-digit number with digits *ab* can be written algebraically as $10a + b$.

Enabling prompt:

- **Resources:** Show the reSolve PowerPoint *1b How Many Blocks?*
  This scaffolds students through the process of writing a general algebraic form for a two-digit number.

Ask students how you would algebraically write the number *ab* with reversed digits.
Work through with students or, if appropriate, ask them to work through the subtraction:

$$\begin{aligned} 10a + b - (10b + a) &= 10a + b - 10b - a \\ &= 9a - 9b \\ &= 9(a - b) \end{aligned}$$

Teacher notes:

- If the number is $10a + b$, then the sum of the digits is $a + b$.
- $$\begin{aligned} 10a + b - (a + b) &= 10a + b - a - b \\ &= 9a \end{aligned}$$
- So, the result is always nine times the tens digit of the original number.

Three-digit numbers

Ask students to use algebra to predict what will happen when a similar 'reverse the digits and subtract' task is used with a three-digit number.

First, ensure that students see that a general form of a three-digit number is $100a + 10b + c$.

Teacher note:

- $$\begin{aligned} 100a + 10b + c - (100c + 10b + a) &= 100a + 10b + c - 100a - 10b - c \\ &= 99a - 99c \\ &= 99(a - c) \end{aligned}$$

We can predict that the answer will be 99 times the difference between the hundreds digit and the units digit. This means that the answer will be divisible by both 9 and 11.

Students can test their prediction with some three-digit numbers.

Ask students what must be true about the original number to make the trick 'interesting'.

- The hundreds and units digits must be different, otherwise the answer will be zero (which is still divisible by 9 and 11, but is not very interesting). Note that the tens digit in the original number is irrelevant to the result.

T **Teacher note:**

- It is strongly recommended that you do the algebra *first*, as this allows students to use algebra to make a prediction. The sequence of activities is carefully constructed to begin with a teacher-directed explanation that introduces the required algebraic notation, progresses through an activity in which students use a very similar process to justify a slightly different result, and then leads to a third activity that extends the previous two whereby students start with using the algebra to predict what will happen with numbers.

Further activities

Activity 1: Extending the reversing digits tasks

Ask students to write down a two-digit number with different digits, reverse the digits, then subtract the smaller number from the larger number. They will already know that the result is a multiple of 9.

Ask the students to take their answer, reverse the digits, then subtract again. Repeat this process with each answer until only a single-digit number remains.

Pose the questions: *What is the number?*
Can you predict in advance how many steps it will take to reach this number?

The number 9 is always obtained. The difference between the tens digit and the units digit determines the number of steps it takes to reach 9. The table below shows the process.

Difference between digits	Process	Number of steps
1	# → 9	1
2	# → 18 → 63 → 27 → 45 → 9	5
3	# → 27 → 45 → 9	3
4	# → 36 → 27 → 45 → 9	4
5	# → 45 → 9	2
6	# → 54 → 9	2
7	# → 63 → 27 → 45 → 9	4
8	# → 72 → 45 → 9	3
9	# → 81 → 63 → 27 → 45 → 9	1

Repeat the process with a new number (i.e. reverse the digits, then subtract). Take the answer, reverse the digits, then add the two numbers together. If the answer is 9, use 09.

Pose the questions: *What happens? Can you explain why?*

The result is always 99. The only numbers resulting after the first subtraction are 09 or 90, 18 or 81, 27 or 72, 36 or 63, and 45 or 54. In each case, reversing the digits and then adding gives 99.

Repeat the same processes for a three-digit number.

Continued reversing and subtracting always produces 99. The difference between the hundreds and units digits determines the number of steps. The results are the same as in the previous table for two-digit numbers.

Reversing and subtracting followed by reversing and adding always produces 1089.

T

Teacher note:

- This last result can be made to look very impressive by writing the number 1089 in soap on the back of your hand. Ask a student to write their answer on a piece of paper, burn it and rub the ashes on your hand. The number 1089 will 'magically' appear.

Activity 2: Some more mind reading!

Ask students to select three different digits and use them to make two three-digit numbers, whereby no digit can be used for the same place value in both numbers. For example, a student choosing 1, 4 and 6 could make 164 and 641. Ask students to subtract the smaller number from the larger number.

Choose a student to read out only two digits of their result, and the teacher will be able to miraculously identify the digit that the student left out. Using the previous example of 164 and 614, they subtract to give 477. Of the three digits, the student might read out the two digits 7 and 7, in which case the teacher can say that the digit 4 was left out. Or the student might read out the two digits 4 and 7, in which case the teacher can say another 7 was left out.

Repeat this process a number of times and discuss with students how the mind reading works.

T

Teacher note:

- After a while, students will notice that the sum of the numbers in their answer is always 9 or 18. If they don't notice, draw this to their attention! This makes it easy to work out the number omitted, unless it is a zero or a 9.
- Let the three digits be a, b and c. Then the possible numbers that can be made are:
  - ◊ $100a + 10b + c$
  - ◊ $100a + 10c + b$
  - ◊ $100b + 10a + c$
  - ◊ $100b + 10c + a$
  - ◊ $100c + 10a + b$
  - ◊ $100c + 10b + a$
- When any two of the expressions above are subtracted from each other, the coefficient of each variable a, b and c will be 0, 9, 90, 99, 990 or 999. The resulting expression involving a, b and c will always have a common factor of 9, so the number is divisible by 9.
- Any number divisible by 9 has a digit sum that is a multiple of 9. In this case, the digit sum is either 9 or 18 since it is a one-, two- or three-digit number and cannot be zero or 999.

LESSON 2

re(Solve) MATHS BY INQUIRY

Tens and Units

Reverse and Add

Y8

About this lesson

This is one of two lessons that use algebra to explain results in arithmetic by expressing two- and three-digit numbers in the general form $10a + b$ or $100a + 10b + c$. Students start by reversing the digits of a two-digit number, adding it to the original number and observing that the result is always a multiple of 11. Connections are made to statistical concepts by randomly generating a large sample of two-digit numbers.

Australian Curriculum: Mathematics (Year 8)

ACMSP206: Explore the practicalities and implications of obtaining data through sampling using a variety of investigative processes.

Mathematical purpose

- The resource develops students' understanding of algebra as generalised arithmetic. Students learn to express two- and three-digit numbers in the general form $10a + b$ or $100a + 10b + c$, and use this knowledge to explain the results of arithmetic operations involving numbers with their digits reversed. The resource links the ideas of place value, learned in the primary years, with algebraic reasoning, and revisits ideas from statistics by analysing randomly generated two-digit numbers.

Learning intention

- To observe what happens when performing operations that involve numbers with reversed digits. To use technology to analyse the solutions we get from a simple procedure.

Time

Two lessons of approximately 1 hour each.

Vocabulary

- dot plots

Resources

- reSolve Excel Spreadsheet *2a Reverse and Add*
- reSolve PowerPoint *1b How Many Blocks?*
- Tinkerplots application and reSolve Tinkerplots file *Reverse and Add* (optional)
- a 10-sided dice

reSolve: Mathematics by Inquiry is funded by the Australian Government and is managed by the Australian Academy of Science.

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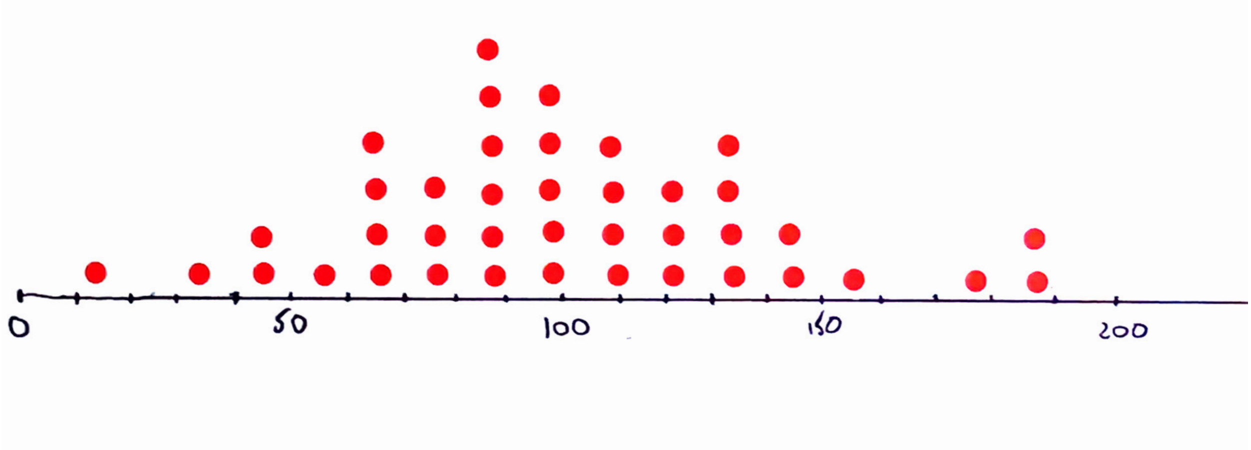
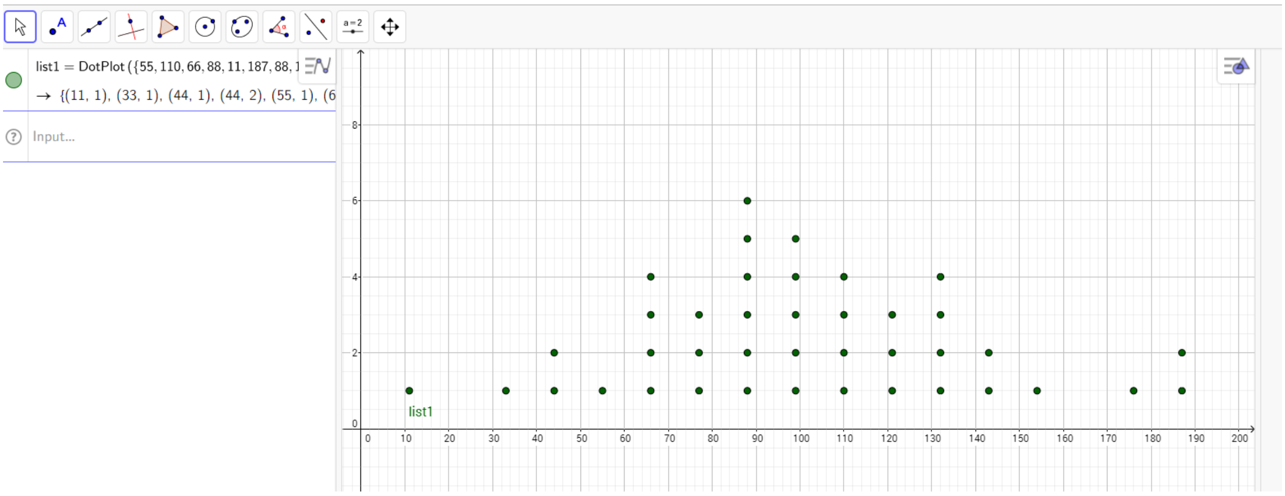
Reverse and add

Ask students to write down a two-digit number, then reverse the digits and add the two numbers. The digits can be the same and can include zero (where numbers such as 9 are written as 09). Repeat and tally the results. This might be by:

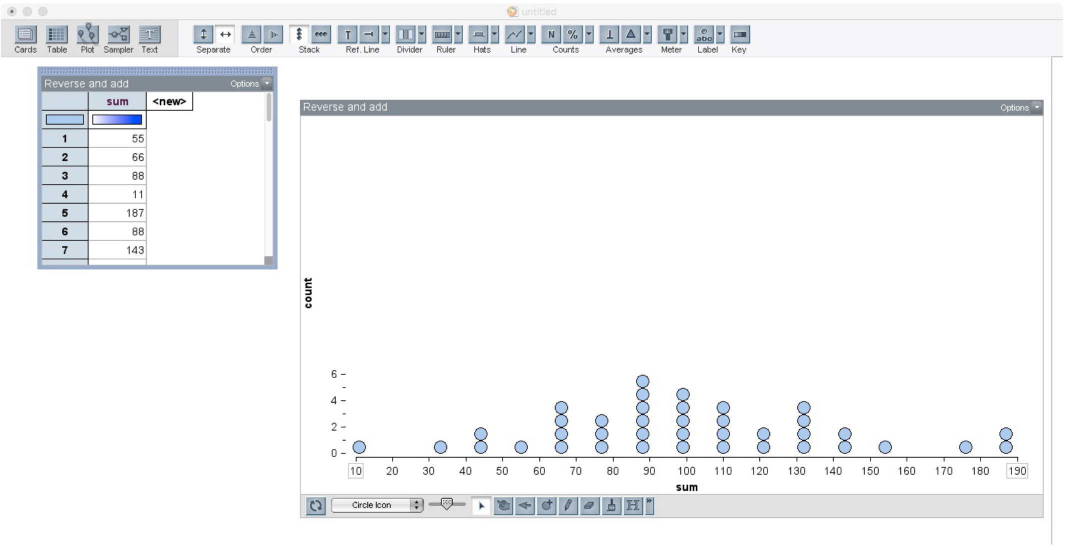
- asking students to put dots onto a number line to create a class dot plot such as that shown below.



- using the GeoGebra graphing calculator Dot Plot command and entering the class results into a list. For example, entering list1 = DotPlot({55,110,66,88,11,187,88,143,110,99,44,88,88,33,44...}) returns a set of points that are then plotted onto the graph, in this case (11, 1), (33, 1), (44, 1), (44, 2) and so on, showing that 44 appears twice in the list.



- using statistical software such as Tinkerplots.



- entering the results into a list on a graphics calculator.

Ask students to share their observations.

Possible student responses:

- The only possible sums are multiples of 11.
- The result is always 11 times the sum of the two digits.
- Multiples of 11 around 100 are more common than others.

Explaining the result algebraically

If students have completed Lesson 1: Mathematical Mind Reader, they ought to be able to use algebra to show why the result is always 11 times the sum of the digits.

If not, discuss with students how a two-digit number with digits *ab* can be written algebraically as $10a + b$.

Enabling prompt:

- **Resources:** Show the reSolve PowerPoint *1b How Many Blocks?* This scaffolds students through the process of writing a general algebraic form for a two-digit number.

Ask students how you would algebraically write the number *ab* with reversed digits.
Work through with students or, if appropriate, ask them to work through the addition:

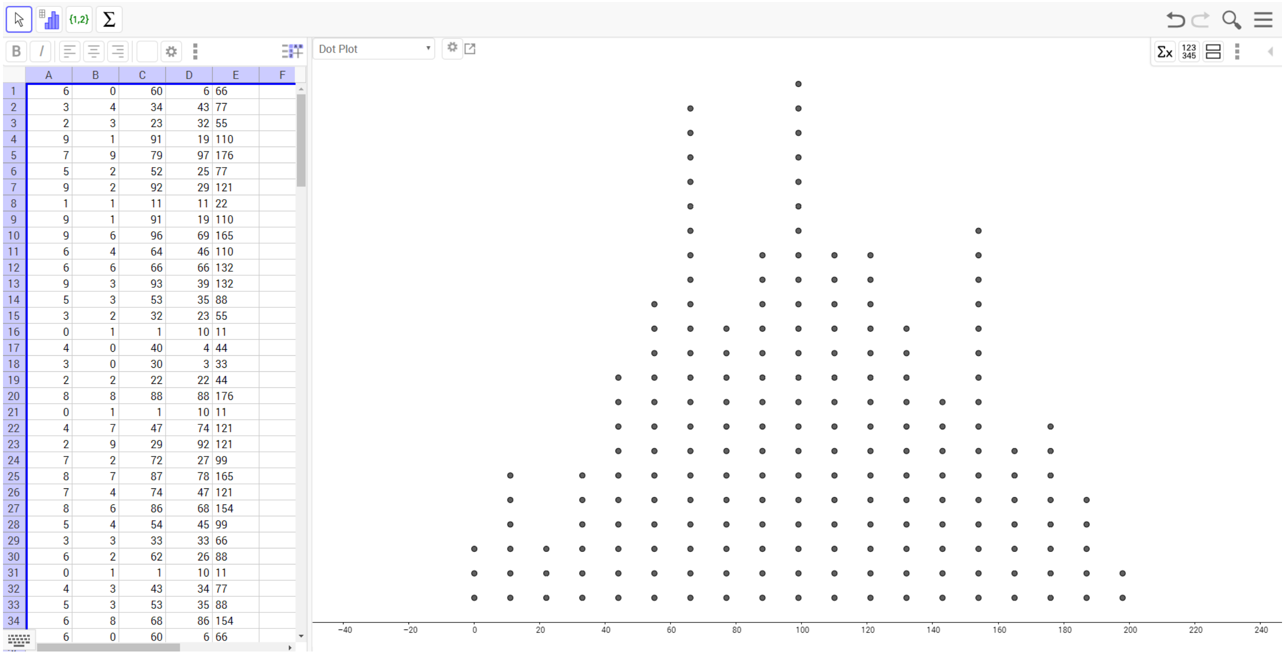
$$\begin{aligned} 10a + b + (10b + a) &= 10a + b + 10b + a \\ &= 11a + 11b \\ &= 11(a + b) \end{aligned}$$

Distribution of results

Depending on the digits chosen by the students, the distribution of results may be centred around the numbers 88 and 99. To appreciate the distribution of results more accurately, we need a larger sample of results generated using a random number generator.

A 10-sided dice is a suitable random number generator if you wish to generate results by hand. Alternatively, there are many ways to randomly generate results using technology, including:

- using the spreadsheet function in GeoGebra together with one-variable analysis. The following image shows the result of generating 200 pairs of random digits between zero and 9, creating a two-digit number, then reversing the digits and adding.



	A	B	C	D	E	F
1	6	0	60	6	66	
2	3	4	34	43	77	
3	2	3	23	32	55	
4	9	1	91	19	110	
5	7	9	79	97	176	
6	5	2	52	25	77	
7	9	2	92	29	121	
8	1	1	11	11	22	
9	9	1	91	19	110	
10	9	6	96	69	165	
11	6	4	64	46	110	
12	6	6	66	66	132	
13	9	3	93	39	132	
14	5	3	53	35	88	
15	3	2	32	23	55	
16	0	1	1	10	11	
17	4	0	40	4	44	
18	3	0	30	3	33	
19	2	2	22	22	44	
20	8	8	88	88	176	
21	0	1	1	10	11	
22	4	7	47	74	121	
23	2	9	29	92	121	
24	7	2	72	27	99	
25	8	7	87	78	165	
26	7	4	74	47	121	
27	8	6	86	68	154	
28	5	4	54	45	99	
29	3	3	33	33	66	
30	6	2	62	26	88	
31	0	1	1	10	11	
32	4	3	43	34	77	
33	5	3	53	35	88	
34	6	8	68	86	154	
	6	0	60	6	66	

  ◊ The relevant instructions are:
    1. Cells A1 and B1 each contain the instruction RandomBetween(0,9). These are then filled down.
    2. Cells C1 and D1 contain the instructions =10*A1+B1 and =10*B1+C1, respectively. These are then filled down.
    3. Cell E1 contains the instruction =C1+D1, which is then filled down.
    4. Highlight column E and click on the image of the histogram in the top left corner (One-Variable Analysis).
    5. In the drop down menu at the top left of the graphics view, select Dot Plot.

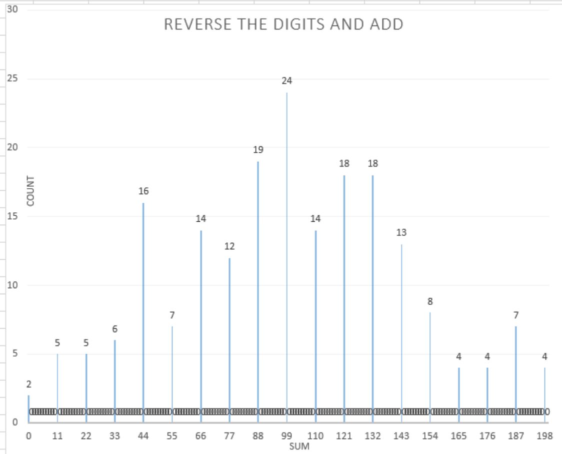
  ◊ The data can be recalculated by pressing F9 on your keyboard to generate new random digits.

  ◊ This GeoGebra workbook can be accessed at https://www.geogebra.org/classic/ftQNtwzB.

- using a spreadsheet to generate data.

Resources: The following image is from reSolve Excel Spreadsheet *2a Reverse and Add*.

	A	B	C	D	E	F	G
1	Tens and Units 2: Reverse and add						
2	*Generate two random digits*		*Make and add the two numbers*			*How many of each?*	
3	Digit 1	Digit 2	Number	Reverse digits	Sum	Sum	Count
4	8	1	81	18	99	0	2
5	4	6	46	64	110	1	0
6	2	2	22	22	44	2	0
7	9	3	93	39	132	3	0
8	8	2	82	28	110	4	0
9	2	0	20	2	22	5	0
10	0	3	3	30	33	6	0
11	9	9	99	99	198	7	0
12	7	1	71	17	88	8	0
13	5	7	57	75	132	9	0
14	4	9	49	94	143	10	0
15	5	4	54	45	99	11	5
16	7	3	73	37	110	12	0
17	1	7	17	71	88	13	0
18	3	1	31	13	44	14	0
19	9	8	98	89	187	15	0
20	9	7	97	79	176	16	0
21	3	1	31	13	44	17	0
22	0	5	5	50	55	18	0
23	3	8	38	83	121	19	0
24	8	6	86	68	154	20	0
25	9	8	98	89	187	21	0
26	5	8	58	85	143	22	5
27	3	7	37	73	110	23	0
28	5	4	54	45	99	24	0
29	0	1	1	10	11	25	0
30	3	9	39	93	132	26	0
31	0	7	7	70	77	27	0
32	2	4	24	42	66	28	0



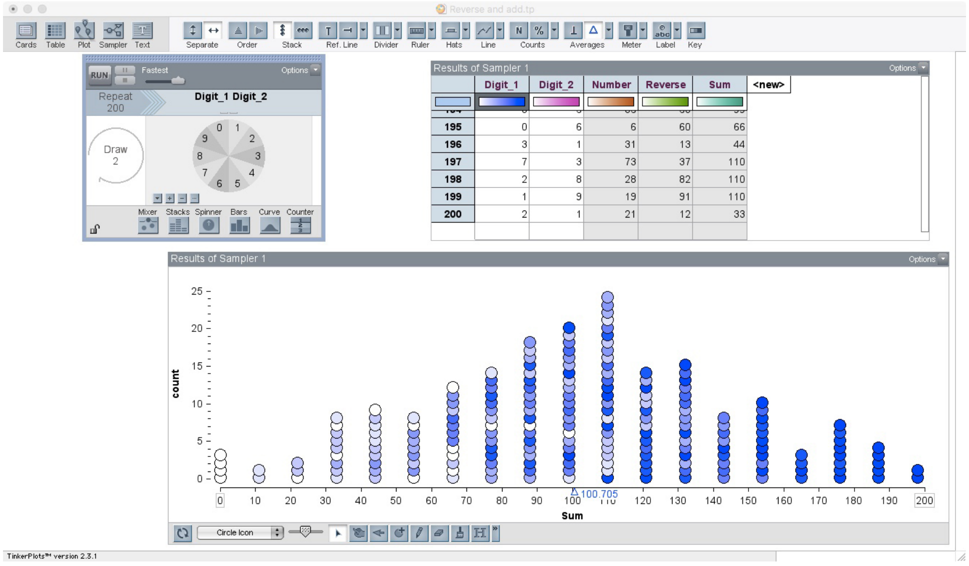
◊ The formulas used are:

A	B	C	D	E	F	G
Digit 1	Digit 2	Number	Reverse digits	Sum	Sum	Count
=RANDBETWEEN(0,9)	=RANDBETWEEN(0,9)	=10*A4+B4	=10*B4+A4	=C4+D4	0	=COUNTIF(E$4:E$203,F4)
=RANDBETWEEN(0,9)	=RANDBETWEEN(0,9)	=10*A5+B5	=10*B5+A5	=C5+D5	=F4+1	=COUNTIF(E$4:E$203,F5)

- Columns A to E generate random digits and sum the two-digit numbers. Column F is a list of numbers from zero to 200. Column G counts the number of times the number in column F appears in column E. The only numbers that will occur once or more are multiples of 11.

◊ A new set of results can be generated by pressing F9 on your keyboard.

- using Sampler in Tinkerplots to randomly generate numbers. This example uses the same formulas to generate the number, its reverse and their sum. By changing Repeat 200 to Repeat 500, 1000, 10 000 and showing the position and value of the mean, the effect of increasing the sample size can be seen very easily. The following image is of a sample of 200 numbers.



◊ **Resources:** The reSolve Tinkerplots file *2b Reverse and Add* can be used if you have the Tinkerplots application on your computer.

Discussion

Discuss with students why sums such as 88, 99 and 110 occur more frequently than zero or 198.

Discussion points might include:

- The result is 11 times the sum of the digits.
- There is only one way to get a digit sum of zero or 18, which will give zero and 198, respectively. These are two zeros or two 9s.
- A digit sum of 1 or 17 can be obtained in two ways each: (0, 1) or (1, 0) and (9, 8) or (8, 9).
- A digit sum of 9 (giving 99) can be obtained in 10 ways: (0, 9), (1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1) or (9, 0).
- The following table shows the number of ways of obtaining each digit sum.

Digit sum	Result	Number of ways to obtain digit sum
0 or 18	0 or 198	1
1 or 17	11 or 187	2
2 or 16	22 or 176	3
3 or 15	33 or 165	4
4 or 14	44 or 154	5
5 or 13	55 or 143	6
6 or 12	66 or 132	7
7 or 11	77 or 121	8
8 or 10	88 or 110	9
9	99	10

- When a larger sample size is used, the mean varies less from the expected result (i.e. 99) than when a smaller sample size is used.

Pose the question: *Can this strategy be extended to bigger numbers?*

Ask students to write down a three-digit number, reverse the digits, then add it to the original number. Collect some answers. Ask students to explain why there is a large number of possible answers with little apparent relationship between them.

Teacher notes:

- If the original number is $100a + 10b + c$, then the number with its digits reversed is $100c + 10b + a$. Adding these gives $101(a + c) + 20b$.
- If we allow zeros for a and c, there are 19 possible sums for $a + c$; and for each of these there are 10 possible values of b. So, there are $19 \times 10 = 190$ different results.
- It is worth noting, however, that $101(a + c)$ results in a palindromic number when $a + c$ is between 1 and 9 inclusive, and the three-digit number remains a palindrome when b is 4 or less.
- For four-digit numbers $1000a + 100b + 10c + d$, the result is $1001(a + d) + 110(b + c) = 11(91(a + d) + 10(b + c))$. So for four-digit numbers the result is always divisible by 11.
- For any number with an even number of digits, both the sum of the number and the sum of its reverse are always divisible by 11.

Further activities

Activity 1: Two from three

Ask a student to think of three different digits and to write down (where everyone can see) all possible two-digit numbers that can be made by selecting two from the three digits so that no digit is repeated. For example, the digits 2, 3, 5 can make the numbers 25, 52, 35, 53, 23 and 32.

While the student is writing all the possibilities, announce your prediction of the sum of the list of two-digit numbers (in this case it is 220). Let the class check that your prediction is correct.

- There should be a total of six numbers in the list. The sum will be 22 times the sum of the three digits selected, which can be calculated quickly by doubling and multiplying by 11.

Assuming that the students now have considerable experience and practice at writing two-digit numbers as $10a + b$, ask them to explain how the sum could be found so quickly.

- They should observe that each digit occurs twice in the tens column and twice in the units column. Hence, the sum will be $20(a + b + c) + 2(a + b + c) = 22(a + b + c)$, which is 22 times the sum of the digits.

Activity 2: Two from four or more

Repeat Activity 1 but this time use four digits and choose two of them to make all possible two-digit numbers. Also try the activity using five or six digits.

- For four digits there will be 12 numbers, with the sum being $33(a + b + c)$.
- For five digits there will be 20 numbers, with the sum being $44(a + b + c)$.
- For n digits (where $n \leqslant 9$) there will be $n(n - 1)$ numbers, with the sum being $(n - 1)(a + b + c)$.