re(Solve) MATHS BY INQUIRY

# Sums and Differences of Squares

Y10

## Summary of learning goals

- Students build their skills in using the special algebraic results relating to perfect squares and differences of squares. They use algebraic reasoning to justify arithmetic results and make connections to visualisations that involve areas.

### Australian Curriculum: Mathematics (Year 10)

**ACMNA233:** Expand binomial products and factorise monic quadratic expressions using a variety of strategies.

- Using the identities for perfect squares and the difference of squares to factorise quadratic expressions.

## Summary of lessons

### Who is this sequence for?

- This set of three lessons is designed to deepen and consolidate students' understanding of the algebraic identities $a^2 - b^2 = (a + b)(a - b)$ and $(a + b)^2 = a^2 + 2ab + b^2$. It is assumed that students have already encountered these identities in their initial learning of the expansion and factorisation of binomial products. The lessons may be done in any order.

### Lesson 1: Quarter Squares

A historical calculation method is used to show students an alternative method for multiplying two-digit numbers. After exploring and becoming familiar with the method, students use algebraic skills, in particular the binomial expansion of perfect squares, to justify why the method always works. Several sustaining activities, including a visual method for showing the identity used in the multiplication, are included.

### Lesson 2: Filling Corners

Students use a visual method for transforming a rectangle into a square by dissecting and rearranging the rectangle, then filling in the missing corner with a square. This introduces a reasonably efficient method for finding pairs of factors and, hence, testing for primality, and the method is trialled using a spreadsheet and Python code. Students use the algebra of the difference of two squares to show why the method works. Some related activities sustain the learning.

### Lesson 3: Algebraic Allsorts

Students engage in a range of activities, including visualisation, methods for rapid calculation, and solving word problems that rely on the difference of two squares or the binomial expansion of perfect squares. Several of these activities rely on atypical problem-solving wherein students look at problems holistically (e.g. using unknown values, but *without solving for the unknown values*).

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# Reflection on this sequence

## Rationale

Approaching algebra as generalised arithmetic shows students the power of algebra in abstracting number. This focus on algebra as generalised arithmetic is typically under-represented in high school mathematics in favour of more time spent on functions and equations. The first two lessons of this sequence are deliberately designed to have historical and cultural connections in order to provide real-world foundations for this abstract approach.

### reSolve mathematics is purposeful

- By providing students with historical and visual interpretations of the same algebraic concepts, students are encouraged to form connections between mathematics as an abstract discipline, as an endeavour that has evolved over time, and as a way of modelling real-world problems.
- This sequence also utilises technology: students construct spreadsheets to test hypotheses and can code their own algorithms (or work with precoded programs) to explore ways of solving the set tasks.

### reSolve tasks are inclusive and challenging

- Each task in this sequence begins with an initial shared experience that orients students to the task, allowing all students to come into the lesson on an equal level. The challenge of this sequence rests in requiring students to use previously established algebraic knowledge in new and unusual applications and contexts: atypical visual representations, historical methods originally designed for very particular circumstances, and written problems that explicitly avoid routine approaches.

### reSolve classrooms have a knowledge-building culture

- The tasks in this sequence require students to investigate individually and then pool their findings in order to draw conclusions that can be generalised. Rather than setting problems to be solved, each task provides a resource or strategy that students apply to their own questions. Students set their own parameters for their investigation and then justify their findings.

LESSON 1

re(Solve) MATHS BY INQUIRY

# Sums and Differences of Squares

## Quarter Squares

Y10

## About this lesson

A historical calculation method is used to show students an alternative method for multiplying two-digit numbers. After exploring and becoming familiar with the method, students use algebraic skills, in particular the binomial expansion of perfect squares, to justify why the method always works. Several sustaining activities, including a visual method for showing the identity used in the multiplication, are included.

## Australian Curriculum: Mathematics (Year 10)

**ACMNA233:** Expand binomial products and factorise monic quadratic expressions using a variety of strategies.

- Using the identities for perfect squares and the difference of squares to factorise quadratic expressions.

## Mathematical purpose

- Students examine an interesting historical procedure for multiplying numbers without directly using the multiplication strategies we now use. They use algebra to justify that this method will always work. The resource makes connections between mathematical ideas from arithmetic and algebra; uses visual, algebraic and technological methods; and highlights mathematics as a historical and cultural activity.

## Learning intention

- To explore a historical method for long multiplication and proving how it works.

### Time

Three lessons of approximately 1 hour each.

### Resources

- Quarter Squares Historical Introduction (either one copy per student or display it)
- reSolve Excel Spreadsheet *1a Quarter Squares*

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## Using Quarter Tables

**Resources:** Hand out the Historical introduction to students or display it, and allow students to read through it.

Using Table B of the reSolve Excel Spreadsheet *1a Quarter Squares*, choose any three multiplication problems and use the Quarter Square Table to find the results, using the process outlined in the Historical introduction:

1. Find the sum and difference of the two numbers.
2. Look up the codes for the sum and difference.
3. Find the difference between the two codes.

Discuss with students:

- Table B is part of the actual table appearing in Leslie's book.
- More numbers and, therefore, more codes are required for larger products. For example, Table B cannot be used for the product 117 × 95 because their sum is 212, which is too large for this table. However, Leslie's original table would have given the result, as it included numbers up to 1000.

Trialling three different products with the Quarter Square Table shows that the table seems to give correct answers. However, this does not guarantee that the table will work for any product. We need to show that the Quarter Square Table will always give the correct answer.

## Comparing Tables A and B

**Resources:** Display Table A of the reSolve Excel Spreadsheet *1a Quarter Squares*.

Ask students: *How does Table A differ from Table B?*

- The code for every odd number in Table A ends with 0.25. These are omitted in Table B.

Explain to students that Table B is constructed from Table A.

**Pose the problem:** *Choose a multiplication question that involves one or more odd numbers and check that Tables A and B both give the correct result. Explain why the calculation using Table A will always give a whole number regardless of whether the multiplication involves no, one or two odd numbers.*

- Two even or two odd numbers will give both an even sum and product; hence, no decimals will be involved. One odd and one even number will give both an odd sum and an odd product, each of which will have a code ending in 0.25. When these are subtracted, the result will be a whole number.

Ask students to examine more closely the entries for the numbers 1 to 10 in Table A. How are the codes constructed?

**Enabling prompt:**

- *Look closely at the codes for 10, 20 and 40. What sort of relationship might you notice?*

T **Teacher note:**

- Each code is the square of half the number or, alternatively, each code is generated by squaring the number and then quartering it.

$$C = \left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$$

## Investigating and exploring Quarter Tables

### Working backwards

- *A number's code is 14 641. What is the number?*
  - ◊ $\sqrt{14\,641} = 121$. Hence, the corresponding number is $121 \times 2 = 242$.
- *Another number's code is 7056. What is this number?*
  - ◊ $\sqrt{7056} = 84$. Hence, the corresponding number is $84 \times 2 = 168$.
- *The difference between 14 641 and 7056 is 7585. Which two numbers have a product of 7585?*
  - ◊ The sum of the two numbers (say, $a$ and $b$) is 242 and the difference is 168. Hence, $a + b = 242$ and $a - b = 168$.
  - ◊ Solving this pair of simultaneous equations shows us $a = 205$ and $b = 37$. We can check that $205 \times 37 = 7585$.

### Working backwards

Ask students to show algebraically that if the two numbers to be multiplied are $a$ and $b$, then the corresponding codes are $\frac{(a+b)^2}{4}$ and $\frac{(a-b)^2}{4}$. Hence, show that the difference between the two codes is $ab$.

- The sum of $a$ and $b$ is $a + b$. Since each code is the square of half the number, the code corresponding to $a + b$ is $\left(\frac{a+b}{2}\right)^2 = \frac{(a+b)^2}{4}$. Similarly, the code for the difference is $\frac{(a-b)^2}{4}$.
- Proof that the difference between the codes is $ab$:

$$\begin{aligned}\frac{(a+b)^2}{4} - \frac{(a-b)^2}{4} &= \frac{a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)}{4} \\ &= \frac{4ab}{4} \\ &= ab\end{aligned}$$

## Reflection

*Discuss with students:* Why is it important to use algebra to justify why a particular calculation method works?

- Since variables can represent any number, an algebraic justification shows that the technique will always work.

# Further activities

## Activity 1

Create a spreadsheet, similar to Table B, that allows the multiplication of numbers up to 1000.

- The formula required will be =INT((A1/2)^2), where A1 is the number in cell A1.

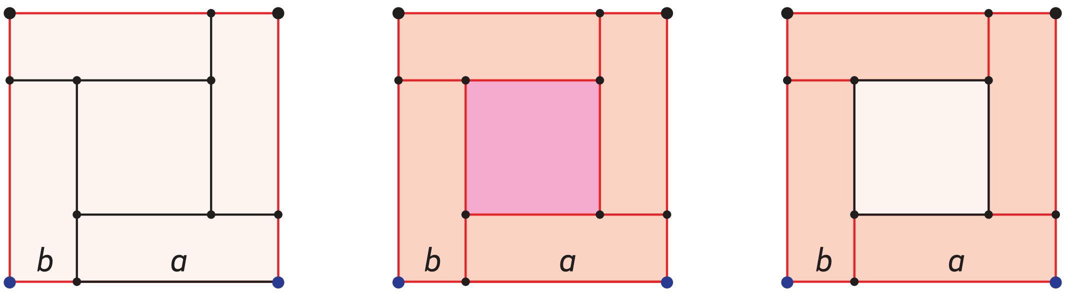
The method outlined in the task can be used together with Table A to multiply two numbers both ending in 0.5. Explain why Table B cannot be used.

- The sum and difference of two numbers both ending in 0.5 will be integers. However, one will always be odd and the other even. Therefore, the exact value of one of the codes (Table A) will end in 0.25, whereas the other will be an integer. Hence, the fractional parts will not cancel out and Table B cannot be used.

## Activity 2

In the diagram below:

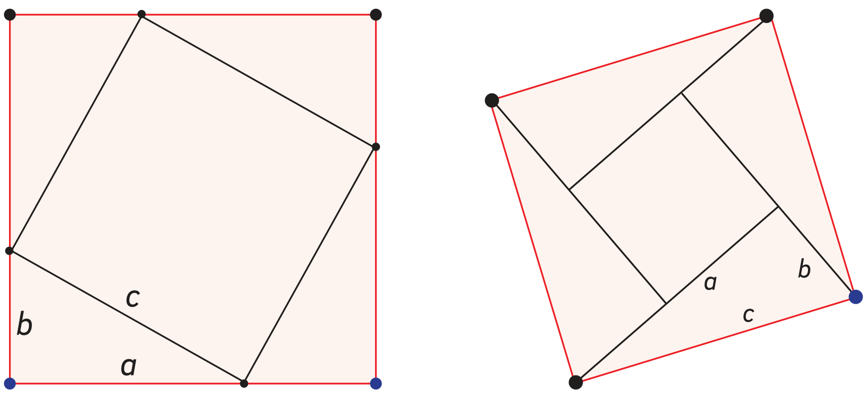
- Explain why the area of the whole square on the left is $(a + b)^2$.
- Explain why the area of the pink square in the middle diagram is $(a - b)^2$.
- Explain why the area of the four orange rectangles on the right is $4ab$.
- Hence, explain how this shows that $(a + b)^2 - (a - b)^2 = 4ab$.




Go to https://ggbm.at/R6gnp5bb and change the value of b to see how the areas change.

## Activity 3

Show that $a^2 + b^2 = c^2$ in each of the two diagrams below. What well-known theorem have you proven?



## Historical introduction: calculating before calculators

In the early 1800s scholars found a way to find the product of two numbers without multiplying them directly. The technique was based on a clever method of multiplication devised by the Babylonians about 4000 years ago.

In 1820 a Scottish mathematician named John Leslie published a book with the wonderful title *The philosophy of arithmetic, exhibiting a progressive view of the theory and practice of calculation, with tables for the multiplication of numbers as far as one thousand*.

In the book, Professor John Leslie explains that the technique he documents was useful as a computational aid to workers who might find themselves in a disadvantaged position because of their educational background.

The extract below is from what Leslie called a Quarter Square Table.

| | |
|---|---|
| 52 | 676 |
| 53 | 702 |
| 54 | 729 |
| ... | ... |
| ... | ... |
| ... | ... |
| 73 | 1332 |
| 74 | 1369 |

To calculate the product of any two numbers (say 63 and 11), the employee should:

1. Find the sum and difference of the two numbers. *In this case, the sum is 74 and the difference is 52*.
2. Look up the codes for 74 and 52. *They are 1369 and 676*.
3. Find the difference between these codes. This is the product of the two original numbers. *The difference is 693, so $63 \times 11 = 693$*.

The multiplication is completed using only addition and subtraction, which is much simpler than multiplication for people with little education.

LESSON 2

re(Solve) MATHS BY INQUIRY

# Sums and Differences of Squares

## Filling Corners

Y10

## About this lesson

Students use a visual method for transforming a rectangle into a square by dissecting and rearranging the rectangle, then filling in the missing corner with a square. This introduces a reasonably efficient method for finding pairs of factors and, hence, testing for primality, and the method is trialled using a spreadsheet and Python code. Students use the algebra of the difference of two squares to show why the method works.

## Australian Curriculum: Mathematics (Year 10)

**ACMNA233:** Expand binomial products and factorise monic quadratic expressions using a variety of strategies.

- Using the identities for perfect squares and the difference of squares to factorise quadratic expressions.

## Mathematical purpose

- This resource uses the identity $a^2 - b^2 = (a + b)(a - b)$ to develop and explain a method for finding factors of an odd number and, hence, testing for primality. It connects visual and algebraic methods using a technique that easily builds into a visualisation of completing the square. Students make connections between different areas of mathematics, particularly number theory and algebra. The activity is enhanced through the use of spreadsheets and coding.

## Learning intention

- To explore visual representations of algebraic methods and devise a way of finding prime numbers.

### Time

Three lessons of approximately 1 hour each.

### Vocabulary

- composite
- primality

### Resources

- reSolve PowerPoint *2a Filling Corners*
- reSolve Excel Spreadsheet *2b Finding Factors* (optional)

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# Teacher background information

The process used in this task is to subtract the lengths of two sides of an $a \times b$ rectangle and then divide the solution by 2: $\frac{a-b}{2}$. This gives the location where the rectangle is cut. The side lengths of the square that is then used to fill the corner are given by $\left(\frac{a-b}{2}\right)^2$.

The formula used to describe this process is:

$$ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$$

This is the same result used in the related reSolve lesson *Quarter Squares*.

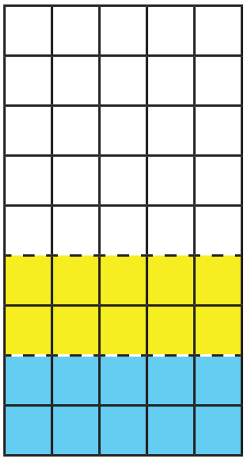
## Fermat's factorisation method

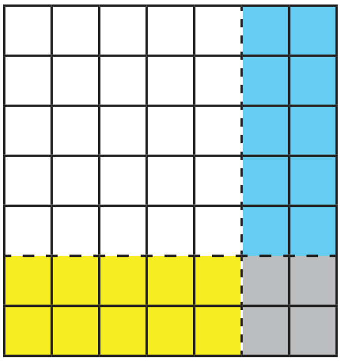
A slightly modified version of the method in this lesson was used by Pierre de Fermat (1607–1665) to find factors of large numbers. Fermat's method started with the observation that the smallest square you need to add to a number to find a pair of factors is not 1, but the difference between the number and the next possible square number. For example, if the number being tested is 1711, the next square number is $42^2 = 1764$. The difference between 1711 and 1764 is 53, so we know that adding 1, 4, 9, 16, 25, 36 or 49 to 1711 cannot produce a perfect square. Hence, the first number worth testing is 64. In fact, adding 225 gives $1936 = 44^2$. Hence, $1711 + 15^2 = 44^2$, so $1711 = (44 - 15)(44 + 15) = 29 \times 59$.

Fermat's method has been improved a number of times. Leonhard Euler (1707–1783) gave a similar but more efficient method that relies on being able to write a number as the sum of two squares in two different ways. See here for more information on Euler's factorisation method.

## Filling the corner

The diagram below shows a rectangle that has been cut and rearranged, then converted into a square by filling in the missing corner with a square. We call this *filling the corner*.

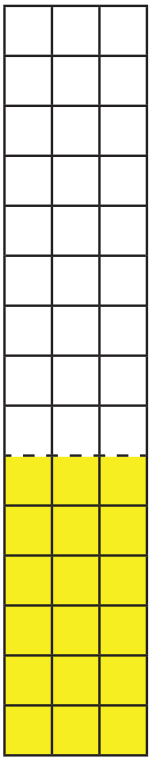


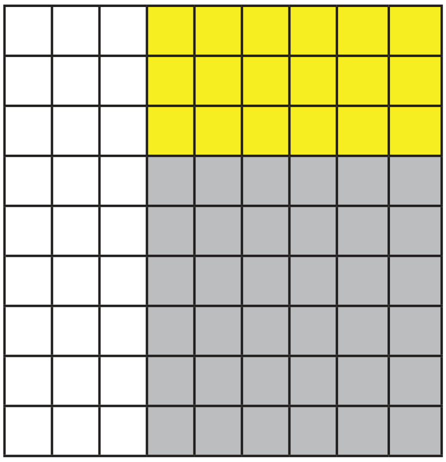


In this case, a $5 \times 9$ rectangle has been converted into a $7 \times 7$ square by being cut, rearranged, and then filling the corner with a $2 \times 2$ square.

How else could you *fill the corner* to convert a rectangle of area 45 into a square?

- You could use a $3 \times 15$ rectangle, cut off and move a $3 \times 6$ rectangle, and fill with a $6 \times 6$ square:





- You could also use a $1 \times 45$ rectangle, cut off and move a $1 \times 22$ rectangle, and fill with a $22 \times 22$ square.
- There are three pairs of factors of 45: 5 and 9, 3 and 15, and 1 and 45. Hence, there are only three ways to convert a rectangle of area 45 into a square.

**Resources:** This process is shown in the reSolve PowerPoint *2a Filling Corners*.

## A class investigation

1. Make rectangles of area 15, 11, 75, 105, and others of your choice. Fill the corners in as many ways as you can. What size squares did you need to use to fill the missing corners?
2. How would you fill the corner of a rectangle with dimensions $a \times b$? What area would the square that is formed have?
3. Explain why you cannot *fill the corner* of a rectangle of area 14 with a square that has integer sides.
   - 14 has factors 2 and 7 or 1 and 14. Both pairs have one even and one odd number. Using the formula given previously, when $a$ is odd and $b$ is even, then $a - b$ and $a + b$ are both odd. Therefore, $\left(\frac{a-b}{2}\right)^2$ and $\left(\frac{a+b}{2}\right)^2$ will not be integers.

Group the rectangles according to the size of the square you used to fill the missing corner. *What do you notice about the areas of the rectangles in each group and their side lengths?*

*For which areas can you never fill the corner with a square that has integer sides?*

- A rectangle that has an area that is a multiple of 2 but not 4 needs a square that has side length ending in 0.5.

*For which areas is there only one way to fill the corner?*

- A rectangle whose area is a prime number can only ever have the corner filled in one way.

## Finding factors

The visual process of *filling the corner* relies on knowing the dimensions of the rectangle. What if the area is known, but the dimensions are unknown? For example, what if we have a very large odd number whose factors we do not know?

Your answers to the Filling the orner task suggests that adding squares might be a way of finding the factors of an odd number, and of testing to see if an odd number is prime. Of course, if the number is even, it is a simple matter to divide by 2 until an odd number is obtained, and then finding the factors of the odd number.

**Resources:** The extract at right is from the reSolve Excel Spreadsheet *2b Finding Factors.*

The spreadsheet can be used to find factors of an odd number and to test whether it is prime. Of course, there is no need to test whether an even number is prime. The spreadsheet can be downloaded and used but we recommend that students create their own version.

The number being tested in this example is 377. The spreadsheet shows that when 64 is added to 377, the result is 441, which is a perfect square. We can write $377 + 8^2 = 21^2$.

| | Finding factors | $n = 377$ | |
|---|---|---|---|
| | Add a square | Sum | Square root |
| 0 | 0 | 377 | 19.4165 |
| 1 | 1 | 378 | 19.4422 |
| 2 | 4 | 381 | 19.5192 |
| 3 | 9 | 386 | 19.6469 |
| 4 | 16 | 393 | 19.8242 |
| 5 | 25 | 402 | 20.0499 |
| 6 | 36 | 413 | 20.3224 |
| 7 | 49 | 426 | 20.6398 |
| 8 | 64 | 441 | 21.0000 |
| 9 | 81 | 458 | 21.4009 |
| 10 | 100 | 477 | 21.8403 |

How can you use this to find factors of 377? Explain how this shows that 377 is composite.

$$\begin{aligned} 377 + 8^2 &= 21^2 \\ 377 &= 21^2 - 8^2 \\ &= (21+8)(21-8) \\ &= 29 \times 13 \end{aligned}$$

- 377 is **composite** because it has factors other than itself and 1.

If we continue to fill down the spreadsheet, another perfect square will eventually be produced. Set up the spreadsheet and fill down until another perfect square results. Write this result arithmetically. What other pair of factors multiplies to 377?

- The next perfect square is at row 188. It shows:

$$\begin{aligned} 377 + 188^2 &= 189^2 \\ 377 &= 189^2 - 188^2 \\ &= (189+188)(189-188) \\ &= 377 \times 1 \end{aligned}$$

Repeat for $n = 97$. Is 97 prime or composite?

- The only integer result for 97 is:

$$\begin{aligned} 97 + 48^2 &= 49^2 \\ 97 &= 49^2 - 48^2 \\ &= (49+48)(49-48) \\ &= 97 \times 1 \end{aligned}$$

- Hence, 97 is prime.

Use your spreadsheet to find all the pairs of factors of 621.

- Integer square roots occur when $2^2$, $30^2$, $102^2$ and $310^2$ are added. The corresponding results are $25^2$, $39^2$, $105^2$ and $311^2$.
- The pairs of factors are the sum and difference of the two numbers in each pair of squares; that is: $27 \times 23$, $69 \times 9$, $207 \times 3$ and $621 \times 1$.

Write a general statement about the result of adding $n^2$ to $2n + 1$. What factors are obtained?

$$\begin{aligned} (2n+1) + n^2 &= (n+1)^2 \\ 2n+1 &= (n+1)^2 - n^2 \\ &= (n+1+n)\big((n+1) - n\big) \\ &= (2n+1) \times 1 \end{aligned}$$

*Discuss with students:* If the original number $(2n + 1)$ is prime, the first time a perfect square is obtained is when $n^2$ is added to the number. This provides a test for primality.

## Exploring the algorithm

Use your spreadsheet to find the pairs of factors of several 'interesting' odd numbers and test them for primality.

A spreadsheet is limited because you might have to fill a long way to find all the pairs of factors and to test whether or not a large number is prime.

We could also write an algorithm:

0. Call the initial number $n$.
1. $a = 1$
2. Find the square root of $n + a^2$.
3. Is this an integer ($b^2$)? If YES, output FACTORS ARE $a + b$ and $a - b$.
4. Increase $a$ by 1.
5. Is $a > (n - 1)/2$? If YES Stop.
6. Go to Step 2.

The Python code at https://trinket.io/python3/01b9eee7fa uses this addition of squares algorithm to find the pairs of factors of an odd number $n$.

The code at https://trinket.io/python3/2eaf9e0ab9 finds pairs of factors by dividing every number from 2 to the square root of $n$.

In what **order** do each of the two methods find pairs of factors? Explain why the addition of squares is often more efficient than the division method (the second link) to determine that a large number is *not* prime.

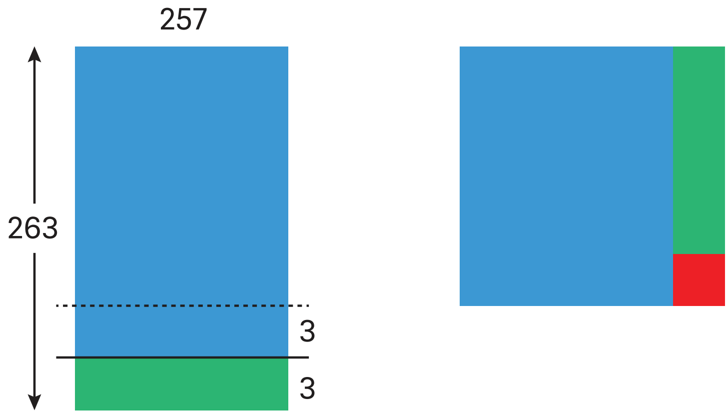
### T Teacher notes:

- The addition of squares method finds pairs of factors that are of similar size (both large) first. This is because if the square being added is $n^2$ and the result is $m^2$, then the two factors are $m - n$ and $m + n$. If these two factors are similar in size, then $n$ must be small and therefore is one of the earlier squares to be added. The division method finds small factors first.
- Large factors are relatively hard to find by conventional methods such as division, but quick to find using the addition of squares method. Hence, the algorithm might find two similar-sized factors quickly, which immediately tells you that the number is not prime. However, it does not tell you if it is prime unless you continue to add squares up to half the number.

Imagine a rectangle of area 67 591. Given that the factors of 67 591 are 257 and 263, both of which are prime, where would you cut the rectangle in order to fill the corner? What size square would you need to fill the corner?

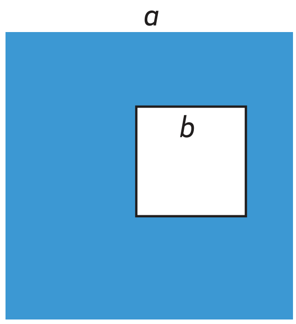
- The rectangle is $257 \times 263$. The difference between 257 and 263 is 6. So you would cut the rectangle along its long side halfway between 257 and 263; that is, at 260. This results in two rectangles: one $257 \times 260$ and the other $257 \times 3$. Moving the $257 \times 3$ rectangle alongside the $257 \times 260$ rectangle gives a square of side length 2609 with a missing corner. Adding a square of side length 3 will give a perfect square of side length 260.
- In this case, $67\,591 + 3^2 = 67\,600 = 260^2$.



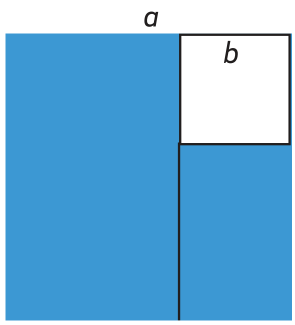

## Further activities

### Activity 1: A visualisation of the difference of two squares

Explain why the shaded area in the diagram below is $a^2 - b^2$.




Now consider the square with side length $b$ moved to the corner of the square of side length $a$.

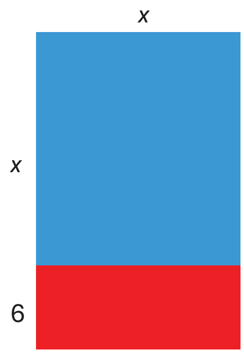



Find the area of the two rectangles and, hence, the total shaded area.
How does this show that $a^2 - b^2 = (a - b)(a + b)$?

## Activity 2: A generalisation of completing the square

Consider the expression $x^2 + 6x$. The diagram below shows how this could be represented as area.




Where would you cut the rectangle so that you could 'fill the corner'?

What size square would you need to 'fill the corner'? What is the size of the resulting square?

- You would cut the red rectangle across the middle (i.e. 3 units from the top and bottom).
- You would then need a $3 \times 3$ square to fill the corner, and the resulting square would have side length $x + 3$.

This is called completing the square and is represented symbolically as $x^2 + 6x + 9 = (x + 3)^2$.

What must be added to complete the square for $x^2 + 10x$?

- $5^2$ or 25

What must be added to complete the square for $x^2 + 5x$?

- $2.5^2$ or 6.25

What must be added to complete the square for $x^2 + 2kx$?

- $k^2$

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### Teacher note:

- If appropriate, this can be used as an introduction to solving quadratic equations by completing the square (ACMNA241). It can then be generalised to derive the quadratic formula for the solution of $ax^2 + bx + c = 0,\ x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

  

## Activity 3: Extension to Mersenne primes

An interesting set of prime numbers are the Mersenne primes. Mersenne primes are of the form $2^n - 1$. In fact, the largest known prime number is a Mersenne prime, currently $2^{74\,207\,281} - 1$. The only Mersenne prime for which $n$ is even is the first Mersenne prime: $2^2 - 1 = 3$.

Explain why $2^n - 1$ is composite for every even value of $n$ greater than 2.

- When $n$ is even, it can be written as $2m$.
- So $2^n - 1 = 2^{2m} - 1 = (2^m - 1)(2^m + 1)$. When $m > 1$, both $2^m - 1$ and $2^m + 1$ are greater than 1; hence, $2^n - 1$ has two factors greater than 1.

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### Teacher notes:

- In fact, $2^n - 1$ is composite for every composite value of $n$. The difference of two cubes (or fifth powers or any odd powers) can always be factorised.
- For example, if $n$ is a multiple of 3, it can be written as $n = 3m$.
- So $2^n - 1 = 2^{3m} - 1 = (2^m - 1)(2^{2m} + 2^m + 1)$. When $m > 1$, both $2^m - 1$ and $2^{2m} + 2^m + 1$ are greater than 1; hence, $2^n - 1$ has two factors greater than 1.
- This shows that if $2^n - 1$ is prime, then $n$ must be prime. It is true that for $n = 2, 3, 5$ and 7, $2^n - 1$ is prime. However, $2^{11} - 1$ is not prime, but $2^{13} - 1$ is. Thus, it is necessary but not sufficient that if $2^n - 1$ is prime, then $n$ is prime. Currently, 49 Mersenne primes have been found. See https://primes.utm.edu/mersenne/index.html for more information, including a list of all known Mersenne primes.

LESSON 2

re(Solve) MATHS BY INQUIRY

# Sums and Differences of Squares

## Algebraic Allsorts

Y10

## About this lesson

Students engage in a range of activities, including visualisation, methods for rapid calculation, and solving word problems that rely on the difference of two squares or the binomial expansion of perfect squares.

## Australian Curriculum: Mathematics (Year 10)

**ACMNA233:** Expand binomial products and factorise monic quadratic expressions, using a variety of strategies.

- Using the identities for perfect squares and the difference of squares to factorise quadratic expressions.

## Mathematical purpose

- Students consolidate their understanding of the algebraic identities relating to the binomial expansion of perfect squares and the difference of two squares by engaging in a variety of related activities, including quick calculations, visualisations and solving number problems. The activities show that algebra can be used to generalise arithmetic results.

## Learning intention

- To review the difference of two squares and the binomial expansion of perfect squares.

### Time

Variable, depending on usage.

**reSolve: Mathematics by Inquiry** is funded by the Australian Government and is managed by the Australian Academy of Science.



Australian Academy of Science

# Activity 1: Quick calculations

## Squaring a number ending in 5 or 0.5

A quick way to square a number ending in 5 is to multiply the tens digit by the tens digit plus 1, then append 25 to the end of the result.

For example, to square 35, multiply 3 by 4 (= 12) and append 25 to the end. The result is, therefore, 1225.

Try this a few more times.

Explain algebraically why it always works.

### Teacher notes:

- Call the original number 10a + 5. Then:

$$\begin{aligned}(10a+5)^2 &= 100a^2 + 100a + 25\\ &= 100a(a+1) + 25\end{aligned}$$

- Note that the result of $100a(a+1)$ will always be a multiple of 100, whereas 25 is smaller than 100. Hence, concatenating the numbers $a(a+1)$ and 25 will give the correct result.

How would you use a similar method to square a fraction ending in 0.5?

- Multiply the whole number part by the whole number part plus 1, then append 0.25.

## Difference of squares

The two calculations below can be done very quickly without a calculator.

- $49 \times 51 = 2499$
- $82 \times 78 = 6396$

Explain how they can be done without using long multiplication.

- $49 \times 51 = (50-1)(50+1) = 50^2 - 1^2 = 2500 - 1 = 2499$
- $82 \times 78 = (80+2)(80-2) = 80^2 - 2^2 = 6400 - 4 = 6396$

What quick calculation could we do using the number 73?

- If we use the number 67, then $73 \times 67 = (70+3)(70-3) = 70^2 - 3^2 = 4900 - 9 = 4891$.

How could you use a similar method to multiply 6.8 by 7.2? Or $4\frac{3}{4} \times 5\frac{1}{4}$?

- $6.8 \times 7.2 = (7-0.2)(7+0.2) = 7^2 - 0.2^2 = 49 - 0.04 = 48.96$
- $4\frac{3}{4} \times 5\frac{1}{4} = (5-\frac{1}{4})(5+\frac{1}{4}) = 5^2 - \left(\frac{1}{4}\right)^2 = 25 - \frac{1}{16} = 25\frac{15}{16}$

LESSON 3

## Activity 2: Working with unknown numbers

The sum of two numbers is 17. The difference between their squares is 31. What is the difference between the two numbers? Why do you not need to find the numbers to solve the problem?

- Let the two numbers be $a$ and $b$.
- We know $a + b = 17$ and $a^2 - a^2 = 31$.
- Since $(a + b)(a - b) = a^2 - b^2$, then:

$$a - b = \frac{a^2 - b^2}{a + b} = \frac{31}{17}$$

Two numbers have a sum of 13. The sums of their squares is 63. What is the product of the two numbers? Why do you not need to know what the numbers are to solve this problem?

- Let the two numbers be $a$ and $b$.
- $a + b = 13$ and $a^2 + b^2 = 63$.
- Since $(a + b)^2 = a^2 + 2ab + b^2$, then:

$$2ab = (a + b)^2 - (a^2 + b^2) = 13^2 - 63 = 169 - 63 = 106$$
$$ab = 53$$

The area of a right-angled triangle is 7 cm$^2$. Its hypotenuse is 6 cm. What is its perimeter?

- Let the two sides meeting at right angles be $a$ and $b$, and call the hypotenuse $c$.
- Then $c^2 = 36 = a^2 + b^2$, using Pythagoras' theorem.
- The area $= \frac{1}{2}ab = 7$; hence, $2ab = 28$.
- Now $a^2 + 2ab + b^2 = (a + b)^2$.
- So $(a + b)^2 = 36 + 28 = 64$.
- So $a + b = 8$; hence, the perimeter $= a + b + c = 6 + 8 = 14$ cm.

If $x^2 + xy = 20$, and $y^2 + xy = 30$, what is $x + y$?

- Add the two equations.
- Then $x^2 + 2xy + y^2 = 50$.
- So $(x + y)^2 = 50$.
- Hence, $x + y = \pm\sqrt{50}$

Suggest some problems of your own that can be solved without finding the values of individual unknowns.

## Activity 3: Square and subtract

Choose any number. Add 1 to your number, then square it. Subtract 1 from your original number, then square it. Subtract your two answers. What do you notice? Explain symbolically and visually why this result will always be true.

- The result is always four times the original number:
  $(n + 1)^2 - (n - 1)^2 = \big((n + 1) + (n - 1)\big)\big((n + 1) - (n - 1)\big) = 2n \times 2 = 4n$
- This could also be illustrated using the expansion of the two perfect squares:
  $(n + 1)^2 - (n - 1)^2 = n^2 + 2n + 1 - n^2 + 2n - 1 = 4n$

Square two consecutive whole numbers and subtract them. What do you notice?
Explain symbolically and visually why this result will always be true.

$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$

- This solution is always odd.

Explain why every odd number can be written as the difference of two square numbers.

- Every odd number can be written as $2n + 1$ or even $(2n + 1) \times 1$.
- Since $2n + 1 = (n + 1) + n$ and $1 = (n + 1) - n$, then $(2n + 1) \times 1 = (n + 1)^2 - n^2$.
- Hence, any odd number can be written as $(n + 1)^2 - n^2$, the difference of two consecutive square numbers.

## Activity 4: Visual proofs

### Part 1

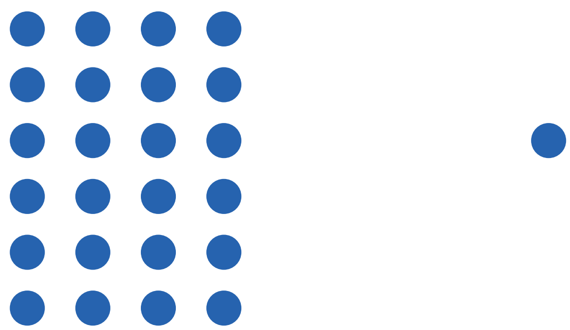
Examine the pattern of equations below.

- $3 \times 5 + 1 = 16$
- $4 \times 6 + 1 = 25$
- $5 \times 7 + 1 = 36$
- ...

Generalise the result symbolically and prove it algebraically.

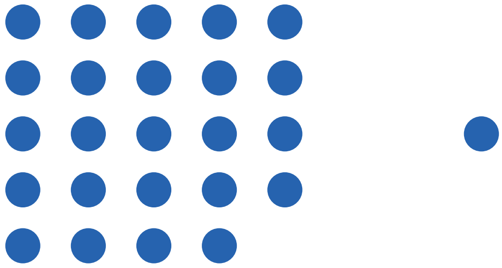
- The result could be written as either:
  - ◊ $n(n + 2) + 1 = n^2 + 2n + 1 = (n + 1)^2$ or
  - ◊ $(n + 1)(n - 1) + 1 = n^2$
- These algebraic proofs use either the expansion of a perfect square or the difference of two squares.

How could you use the diagram below to give a visual proof of the generalisation?



- The rectangle has two more rows than columns (or $n$ columns and $n + 2$ rows). By sliding and rotating the bottom row to form a fifth column and then filling in the gap with the additional dot, we produce a square with one less row and one more column than the original rectangle (i.e. a square with sides of $(n + 1)$).
    - ◊ There is an animation of this process here.



## Part 2

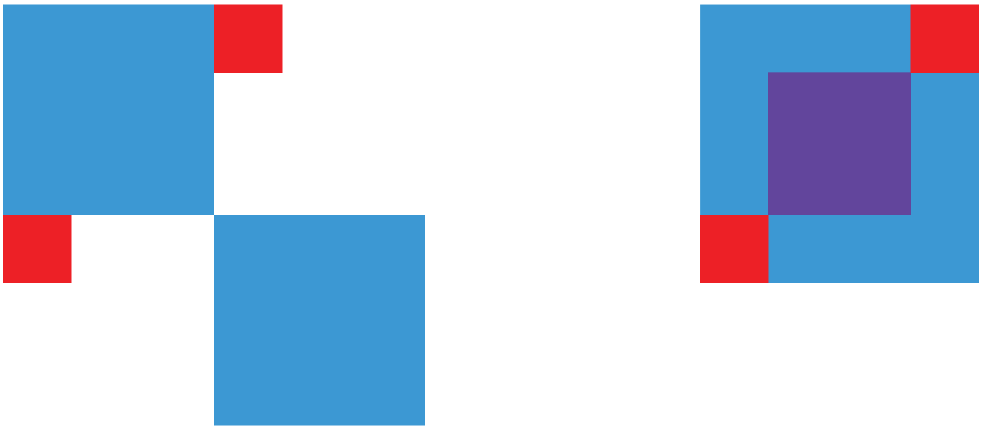
Examine the pattern of equations below.

- $2(3^2 + 7^2) = 10^2 + 4^2$
- $2(5^2 + 1^2) = 6^2 + 4^2$
- $2(11^2 + 3^2) = 14^2 + 8^2$
- ...

Generalise the result and prove it algebraically.

- The generalisation is $2(a^2 + b^2) = (a + b)^2 + (a - b)^2$.
- The right-hand side is $(a + b)^2 + (a - b)^2 = (a^2 + 2ab + b^2) + (a^2 - 2ab + b^2) = 2a^2 + 2b^2$.

Explain how the diagram below proves the result visually.



- The diagram on the left shows two blue squares (call them each area $a^2$) and two red squares (call them each area $b^2$). So it represents $2(a^2 + b^2)$. The diagram on the right shows the blue square at the bottom right moved diagonally to overlap the other blue square. We now have a large square of area $(a + b)^2$, in which the purple square (the overlap, of area $(a - b)^2$) is counted twice.
- So $2(a^2 + b^2) = (a + b)^2 + (a - b)^2$.