re(Solve) MATHS BY INQUIRY

Monte Carlo Simulations

Y9

Summary of learning goals

- Students learn how statistical techniques that rely on randomly generated data can be used to solve problems that have a defined solution that is difficult to solve using traditional methods.

Australian Curriculum: Mathematics (Year 9)

ACMMG216: Calculate areas of composite shapes.

ACMMG222: Investigate Pythagoras' theorem and its application to solving simple problems using right-angled triangles.

ACMSP226: Calculate relative frequencies from given or collected data to estimate probabilities of events involving 'and' or 'or'.

ACMSP283: Compare data displays using mean, median and range to describe and interpret numerical datasets in terms of location (centre) and spread.

Summary of lessons

Who is this sequence for?

- This sequence is for students who are familiar with proportions, area formulae for trapezia, triangles and circles, and Pythagoras' theorem. It links different mathematical concepts through the generation of random points to estimate irregular areas and π.

Lesson 1: Oil Spills

Students compare the efficiency and accuracy of different methods for calculating the area of an irregular shape, using the context of oil spill maps. They are introduced to the concept of the Monte Carlo method for calculating area.

Lesson 2: Estimating π

Students use the Monte Carlo method to estimate the area of a quadrant of a circle. By comparing this with the formula for the area of a circle, students estimate the value of π.

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Reflection on this sequence

Rationale

The sequence of lessons provides an unusual and engaging context that takes advantage of the variation inherent in random data. They tie together ideas from geometry, including areas of irregular shapes, trapezia, triangles and circles, with Pythagoras' theorem and proportional reasoning. Lesson 1: Oil Spills introduces the trigonometric formula for the area of a triangle for those students who are ready; Lesson 2: Estimating π can be used to introduce the distance formula in the Cartesian plane.

reSolve mathematics is purposeful

- This sequence provides interesting historical and environmental applications of Monte Carlo methods in mathematics.

reSolve tasks are inclusive and challenging

- The different contexts will appeal to a wide range of students and show clearly how mathematics is used in real-life problems of interest to students.
- The tasks include a variety of technologies that can be used to generate random numbers and solve the related problems.

reSolve classrooms have a knowledge-building culture

- In Lesson 1: Oil Spills students work collaboratively to compare a variety of methods for calculating area.
- The lessons builds on students' prior knowledge, thus providing an ideal context in which to revise key ideas from measurement and geometry.

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LESSON 1

re(Solve) MATHS BY INQUIRY

Monte Carlo Simulations

Oil Spills

Y9

About this lesson

Students compare the efficiency and accuracy of different methods for calculating the area of an irregular shape, using the context of oil spill maps. They are introduced to the concept of the Monte Carlo method for calculating area.

Australian Curriculum: Mathematics (Year 9)

ACMMG216: Calculate areas of composite shapes.

ACMSP226: Calculate relative frequencies from given or collected data to estimate probabilities of events involving 'and' or 'or'.

ACMSP283: Compare data displays using mean, median and range to describe and interpret numerical datasets in terms of location (centre) and spread.

Mathematical purpose

- To calculate the areas of irregular shapes using a variety of methods.

Learning intention

- To understand the Monte Carlo method for calculating the area of irregular shapes, and to compare its efficiency and accuracy to other methods.

Time

Two lessons of approximately 1 hour each.

Vocabulary

- oil slick
- oil spill
- simulation

Resources

- Student Sheet 1 – Oil Spill Shape (one per student)
- reSolve PowerPoint *1a Oil Spills*
- reSolve Excel Spreadsheet *1b Oil Spill Area*
- reSolve Excel Spreadsheet *1c Monte Carlo Oil Spill Area*
- reSolve image *1d Exxon Valdez*
- reSolve image *1e Deepwater Horizon NASA* (optional)

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Teacher background information

A Monte Carlo simulation uses random sampling to solve problems that are deterministic in nature, but that might be difficult to solve using traditional mathematical methods. The simulation relies on using a large number of randomly generated points to obtain a best estimate. It has widespread applications in areas such as particle physics, telecommunications, meteorology, video game play, financial uncertainty and even law.

In this lesson, students use a simple example of finding an irregular area. They plot randomly generated points in a rectangle of known area that completely encloses the irregular shape. The percentage of points that lie within the shape is therefore an approximation to the ratio of the irregular area to the known area.

Real-world applications of the Monte Carlo method

A very readable account of how the Monte Carlo method was used to solve a real-world problem can be found in the article 'Search for the *SS Central America*: Mathematical Treasure Hunting' (published in 1992 by the journal *Interfaces,* 22: 1). Briefly:

- In 1857 the steamship *SS Central America* sank in a hurricane while en route from Panama to New York. It was carrying gold bars and coins from the California gold rush worth an estimated US$8 million (or nearly US$400 million in modern currency). As well as the Captain, William Herndon, 425 passengers were killed.
- The sinking caused a financial panic and led to the first world financial recession.
- In 1985 the Columbus-America Discovery Group was formed by Thomas Thompson, a mechanical engineer. He put together a team of engineers and mathematicians to search for historic wrecks, including the *SS Central America*.

Lithograph of the 'Wreck of the Central America' by J. Childs (engraver and publisher). Public domain work of art reproduced from http://commons.wikimedia.org/wiki/File:Wreck_of_the_Central_America.jpg

- Lawrence Stone was responsible for creating a probability map to locate the wreck. His team randomly generated 4000 points to represent the best possible position of the *Central America* at the time of the sinking.
- Based on the probability map a search pattern was created, and the wreck was duly found in the summer of 1989. One tonne of gold bars and coins was recovered with an estimated value of US$100–150 million.
- Thirty-nine insurance companies filed a lawsuit, claiming that they had a right to the gold as they had paid compensation in the 19th century. After a legal battle 92% of the gold was awarded to the discovery team.
- Thompson was sued again in 2005 by several of the investors in the project and several of his team members for failing to pay them their due returns. Thompson fled and went into hiding in 2012.
- Thompson was located in Florida in January 2015 and extradited to Ohio, where he remains in prison for criminal contempt until he tells the court where the gold coins are located.

The story is presented in the History Channel documentary *Ship of Gold*.

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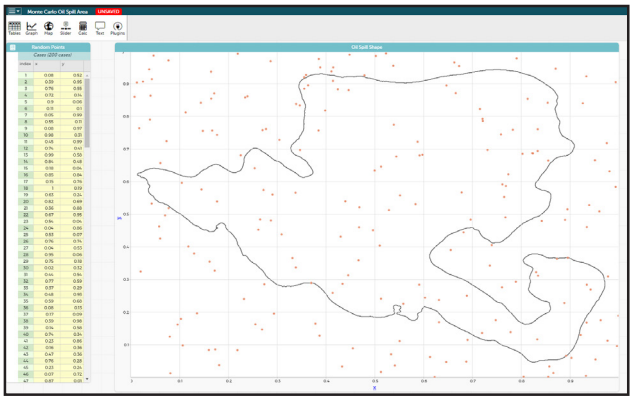
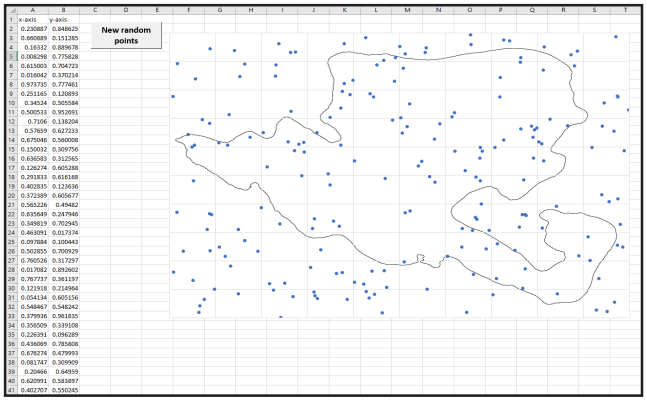
Using the Monte Carlo Oil Spill Area Simulation

There are two options for carrying out the Monte Carlo simulation in this lesson.

Resources: Students can visit https://codap.concord.org/releases/latest/static/dg/en/cert/index.html#shared=50813 to use the online application. Alternatively, reSolve Excel Spreadsheet *1c Monte Carlo Oil Spill Area* can be used.



Online application

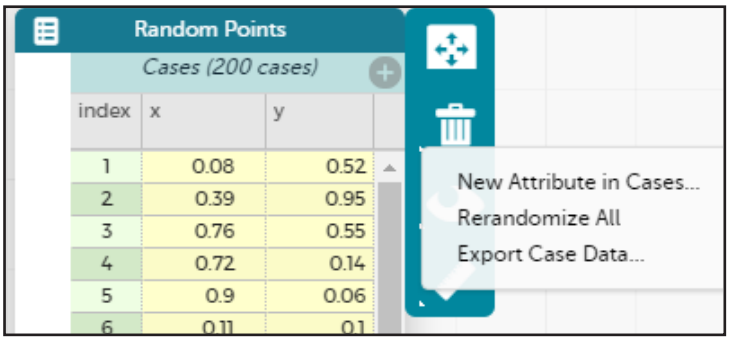


Excel spreadsheet

Both applications work in the same way: the table on the left generates pairs of random numbers between 0 and 1, which are used as x and y coordinates that are plotted over the image. As a non-digital alternative, students can use a printed copy of Student Sheet 1 – Oil Spill Shape and generate the random numbers using their calculators.

To generate a new set of random points in the Excel spreadsheet, press F9 or click on the **New Random Points** button (requires macros enabled).

To generate a new set of random points in the online application, select the **Random** table, then click the ruler icon and choose **Rerandomize All**. To add points in the online application, click in the Index column, choose **Insert Cases...**, and input the number of points to be added. To remove points, select the points to be removed, click the eye icon and choose **Set Aside Selected Cases**.



Area of a triangle using trigonometry

The triangle method for calculating the area of the oil spill requires the formula $A = \frac{1}{2}ab\sin C$.

This formula can be derived as shown below.

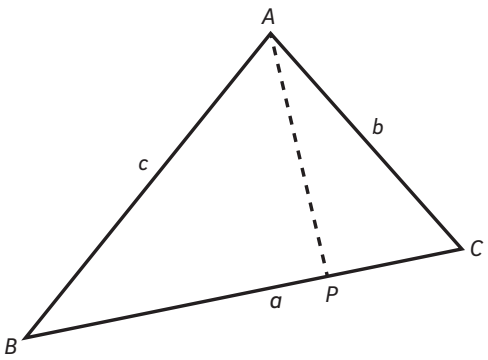
In triangle ABC, draw a perpendicular line, AP, from A to BC.

$$\sin C = \frac{AP}{b}$$

$$\therefore AP = b\sin C$$

$$\therefore \text{Area of triangle} = \frac{1}{2}BC.AP = \frac{1}{2}ab\sin C$$

Note that we have assumed that angle C is acute.
The derivation is still true when C is obtuse, as $\sin C = \sin(180 - C)$.



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Introduction

Resources: Show students slide 2 of reSolve PowerPoint *1a Oil Spills*.

Explain:

On 20 April 2010, there was an explosion on the Deepwater Horizon oil rig while it was drilling for oil from undersea wells. The rig sank but the well continued to gush oil. It took nearly 5 months to seal the well, during which 780 billion litres of oil flowed into the ocean, and the clean-up was not complete until 2014. The spill covered an area containing over 8000 different animal species, which were all affected. The Deepwater Horizon oil spill was one of the largest oil spills in the world and was called the 'worst environmental disaster the US has faced'.

Slide 3 shows a map of the cumulative area covered by the oil spill from 22 April to 1 August 2010. Ask students to estimate the surface area of the spill, taking into consideration the scale of the map.

Slide 4 shows a simplified outline of the oil spill and presents three traditional methods for calculating the area of an irregular shape. Outline each method:

- **Grid method**
  - Draw a grid over the shape and count the number of squares that contain or partially contain the shape.
- **Triangle method**
  - Choose a point near the centre of the shape and draw triangles radiating from this point at a set angle, then calculate the area of each triangle (see Teacher background information).
- **Trapezium method**
  - Divide the shape into trapezia with a set height and calculate the area of each trapezium.

Discuss how to vary the accuracy or complexity of each method; for example:

- Using a grid with larger squares or triangles with larger angles or trapezia with greater heights is quick but gives a very approximate answer.
- Using, for example, 1 mm × 1 mm square grids or triangles with 5° angles increases accuracy but is time-consuming.

Exploration

Resources: Give each student a copy of Student Sheet 1 – Oil Spill Shape.

Allow students to pick one of the three methods and use it to calculate the area of the shape.

Teacher notes:

- Each method has one key variable (i.e. size of grid squares, angle of triangles or height of trapezia). Allow students to independently select a value to use. Later, you will discuss as a class how this has affected their result.
- You may choose to have students repeat the activity using a different method and to compare their answer to the original method used.

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As a class, discuss:

- Who calculated the smallest area? Largest area? Can they justify why their answer might be the smallest/largest?
- What is the class range for the area calculated?
- What is the class's mean area of the shape?
- What are the advantages of the trapezium or triangle method over the grid method?
  - ◊ In particular, each trapezium or triangle requires only two measurements and, since the height or angle is the same in each trapezium or triangle, the calculations can be easily automated using a spreadsheet.

Resources: reSolve Excel Spreadsheet *1b Oil Spill Area* contains examples of spreadsheets for calculating the area using the triangle method and the trapezium method.

Note that Excel's trigonometry functions use radians to measure angles, not degrees. The formula RADIANS(cell) converts degrees to radians.

Monte Carlo method

Show slide 5 of reSolve PowerPoint *1a Oil Spills* and ask students: *Ten points have been randomly placed on the image. How many are within the oil spill? How many are outside the oil spill? What* ***percentage*** *of the points are within the spill? What might that tell us about the area of the spill shape?*

Explain that the Monte Carlo method is a method of problem-solving whereby a computer program generates random numbers and observes the results.

As outlined in the Teacher background information, have students open either the Deepwater Horizon Oil Spill sheet from the reSolve Excel Spreadsheet *1c Monte Carlo Oil Spill Area* or to follow this link: https://codap.concord.org/releases/latest/static/dg/en/cert/index.html#shared=50813.

Outline the following process for students:

1. Calculate the total scaled area of the image, by referring to their original student sheets.
2. Count the number of points that are within the oil spill shape.
3. Calculate the percentage of the 200 points that are within the shape.
4. Use this calculation to find the approximate area of the oil spill.
5. Repeat several times or collate class results and then calculate the mean area of the oil spill.
6. Students might also wish to add more points to the map or remove some and see how this affects their results. See the Teacher background information for directions to do this in the online application.

After students have obtained a mean result with which they are satisfied, as a class discuss how these results compare to their earlier findings using the other three methods. How does the accuracy and the efficiency of the Monte Carlo method compare?

Teacher note:

- The exact area contained within the oil spill shape is 117 554 km^2.

Now that students are familiar with the Monte Carlo method, we will try using it to measure the area of an oil spill for which the previous methods are difficult to apply. Ask students to open the reSolve Excel Spreadsheet *1c Monte Carlo Oil Spill Area* and open the Exxon Valdez Oil Spill sheet or to visit https://codap.concord.org/releases/latest/static/dg/en/cert/index.html#shared=50824. The Exxon Valdez spill (which occurred in 1989) is the second largest oil spill in US waters after the Deepwater Horizon spill. As a class, discuss why the grid/triangle/trapezium methods would be difficult to use with this image, and have students use the Monte Carlo method to calculate the area.

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Teacher notes:

- There are a few ways for students to calculate the total area of the Exxon Valdez image:
  - ◊ **Resources:** Print the image (included with this resource as *1d Exxon Valdez*) and measure its area by hand.
  - ◊ Open the image in an image editing program and count the number of pixels that correspond to the 30 km scale marking (37 pixels), then apply this scale to the 837 pixels × 691 pixels image.
- The official area covered by the Exxon Valdez oil spill is 28 000 km^2.

Reflection

Pose the problem: *Can you use the Monte Carlo simulation to draw an irregular shape that has a particular area?*

- Note that there are multiple strategies that could be used, such as relying on the grid lines or trying to include a certain number of points in the shape, as well as others.

Further activities

Calculating the oil slick area from a satellite photograph

Resources: Challenge students to use the Monte Carlo method to calculate the surface area of the oil slick in image *1e Deepwater Horizon NASA* (a satellite image showing the Deepwater Horizon oil slick on 25 June 2010).

Students will need to use the Monte Carlo simulation to calculate the percentage of points that appear to be within the oil slick, then refer to Google Maps (or similar) to estimate the scale of the image and, hence, find the total area covered by the image.

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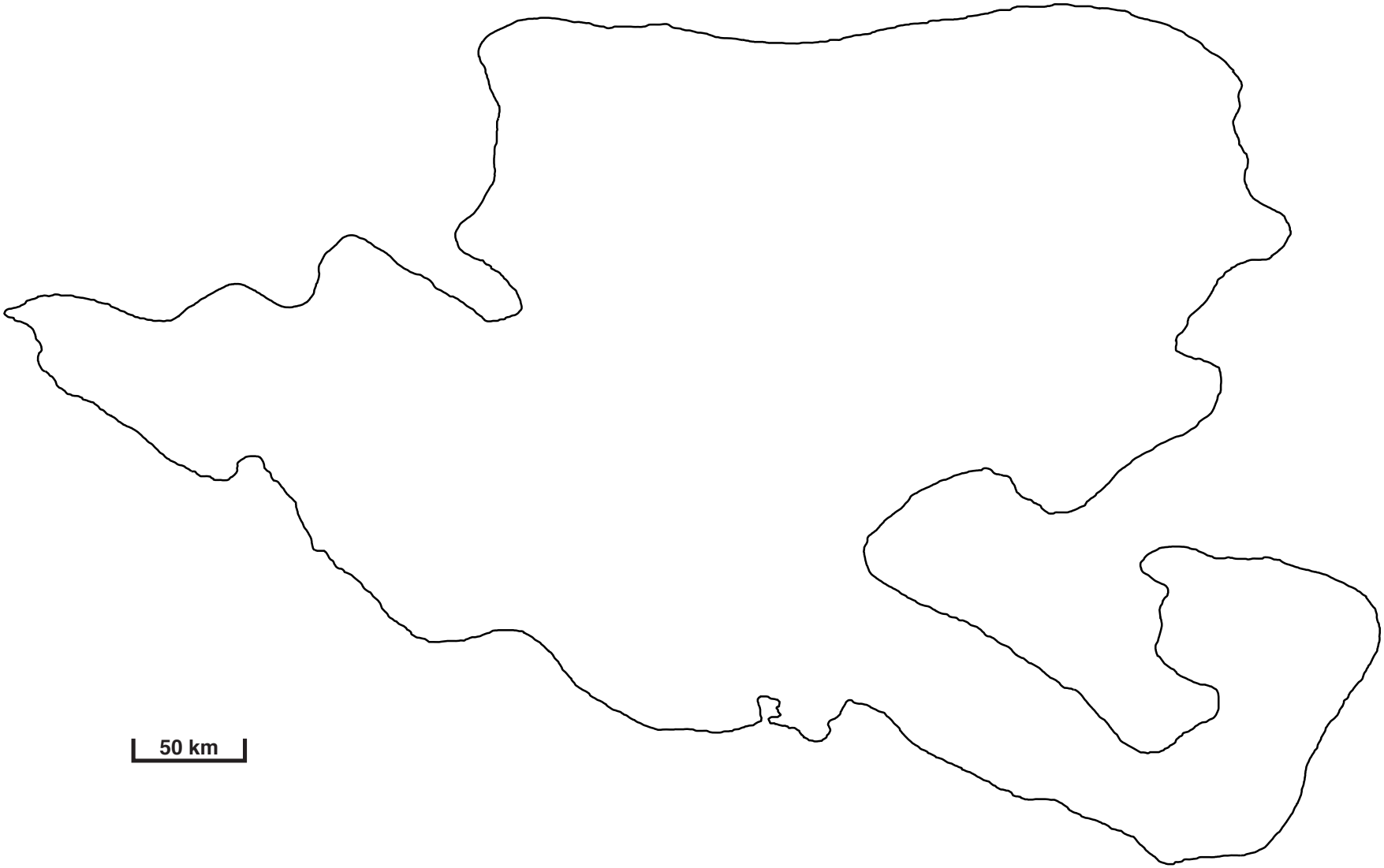
Oil Spill Shape

Name: _____



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LESSON 2

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Monte Carlo Simulations

Estimating π

Y9

About this lesson

Students use the Monte Carlo method to estimate the area of a quadrant of a circle. By comparing this with the formula for the area of a circle, they estimate the value of π.

Australian Curriculum: Mathematics (Year 9)

ACMSP226: Calculate relative frequencies from given or collected data to estimate probabilities of events involving 'and' or 'or'.

ACMMG222: Investigate Pythagoras' theorem and its application to solving simple problems using right-angled triangles.

Mathematical purpose

- Students apply their understanding of Monte Carlo simulations from Lesson 1 to estimate the value of π.

Learning intention

- To discover other interesting applications of the Monte Carlo method.

Time

Two lessons of approximately 1 hour each.

Vocabulary

- Monte Carlo method

Resources

- reSolve PowerPoint *2a Monte Carlo π*
- reSolve PDF *2b Random Circles* (one copy per two students)
  - ◊ Teacher preparation: Print *2b Random Circles* onto A3 paper and cut out the square.
- drawing pins
- reSolve Excel Spreadsheet *2c Circle Spreadsheet*

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A history of π estimations

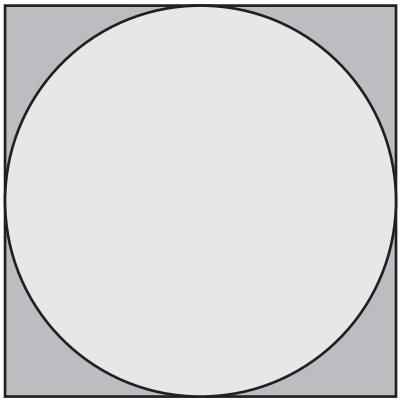
Introduce students to a brief history of π and the accuracy with which it has been estimated or calculated.

Resources: This is given on slide 2 of reSolve PowerPoint *2a Monte Carlo π*:

- 'He made the Sea of cast metal, circular in shape, measuring ten cubits from rim to rim and five cubits high. It took a line of thirty cubits to measure around it' (1 Kings 7:23, approx. 550 BCE). Hence, $\pi \approx 3$.
- The Babylonians (approx. 1800–1600 BCE) gave an approximation of π as $\frac{25}{8}$ (3.125).
- The Rhind Papyrus from ancient Egypt (1650 BCE) provides a formula that gives a value for π of approximately 3.1605.
- In about 1400 CE Madhave of Sangamagrama used the formula $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$ to calculate π to 10 decimal places.
- In 480 CE Zu Chongzhi, a Chinese mathematician and astronomer, used polygons with 24 576 sides to find a value for π of $\frac{355}{113}$.
- Archimedes of Syracuse (287–212 BCE) used polygons and Pythagoras' theorem to show that π is between $\frac{223}{71}$ and $\frac{22}{7}$.
- Georges-Louis Leclerc, Comte de Buffon (1707–1788) proved that if a needle is dropped onto a floor with floorboards the same width as the length of the needle, the probability that a needle will straddle two floorboards is $\frac{2}{\pi}$. This experiment is called Buffon's needle.
  - ◊ An online version is at https://mste.illinois.edu/activity/buffon/#simulation. Ensure that the Needle scale is set to 1. Students could start by dropping 10 needles and counting the number that straddle two lines. If there are n needles straddling the lines, then $\frac{2}{\pi} = \frac{n}{10}$. Hence, $\pi = \frac{20}{n}$. Of course, more needle drops will give more accurate values.
- In 1874 William Shanks took 15 years to calculate π to 707 decimal places, but only the first 527 were correct.
- On 21 January 2019 Emma Haruka Iwao set a new world record by calculating π to 31 415 926 535 897 decimal places. It took 25 machines 121 days to compute. For more information, see: http://www.numberworld.org/digits/Pi/.

The proof of Buffon's result uses integral calculus, but we can use a similar Monte Carlo technique to estimate the value of π.

A physical simulation of π: darts on a board



Show students slide 3 of reSolve PowerPoint *2a Monte Carlo π*, displaying a circle inside a square.

Ask students to calculate the fraction of the square's area that is covered by the circle.

Enabling prompt:

- Assume that the radius of the circle is 1 unit.

Suggest that this image could be used as a dartboard, and ask students to estimate what percentage of darts landing inside the square would also be inside the circle. What assumption is being made? Is this assumption reasonable?

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Resources: To carry out a physical simulation, provide groups of students with a face-down copy of the square cut from reSolve PDF *2b Random Circles*.

Students should not look at the other side of the page.

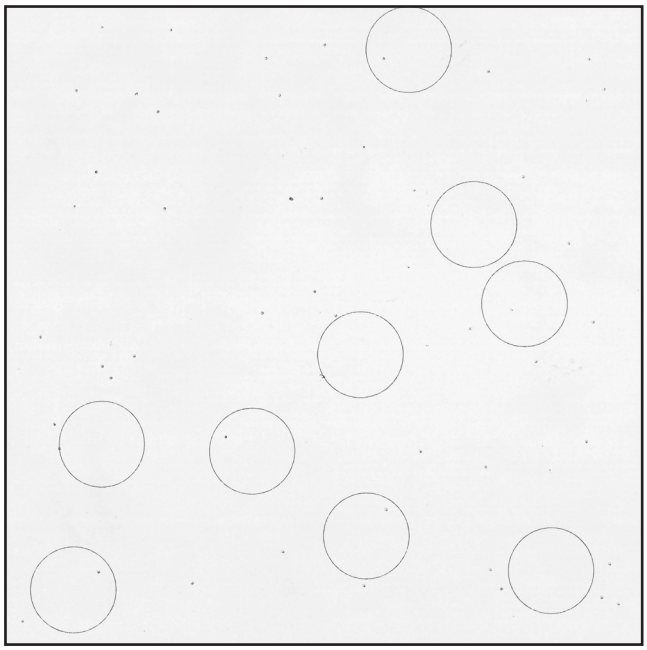
Teacher notes:

- reSolve PDF *2b Random Circles* shows eight circles, each of radius 1 unit, randomly placed without overlaps inside a square of side length 15 units. Using this sheet helps to avoid the issue that throwing darts is not random.
  - ◊ The actual and physical dimensions of the circles and the square are not relevant here, so will not be used.

Have students place the square (still upside down) on the carpet or pin-up board so that they cannot see where the circles are. Students randomly place 50 drawing pins in the paper (or use one pin to make 50 holes), then turn the paper over so that the circles are visible and count the number of holes that are inside a circle.

As per Lesson 1, students can calculate the area of the eight circles from the proportion of pins inside the circles and the total area of the square ($15 \times 15 = 225$ square units). Since each circle has an area of $\pi \times 1^2 = \pi$ square units, the area of the eight circles should be 8π square units. Combining this with the estimate of the area from the Monte Carlo simulation gives an approximate value for π.

In the example below and as shown on slide 4, five points are clearly inside the circles. Two are on the edge of a circle, so we will count that as one inside. This means that six of the 50 points are inside the circles. Hence, $\frac{8\pi}{225} = \frac{6}{50}$ and our estimate for π is $\frac{6 \times 225}{8 \times 50} = 3.375$.



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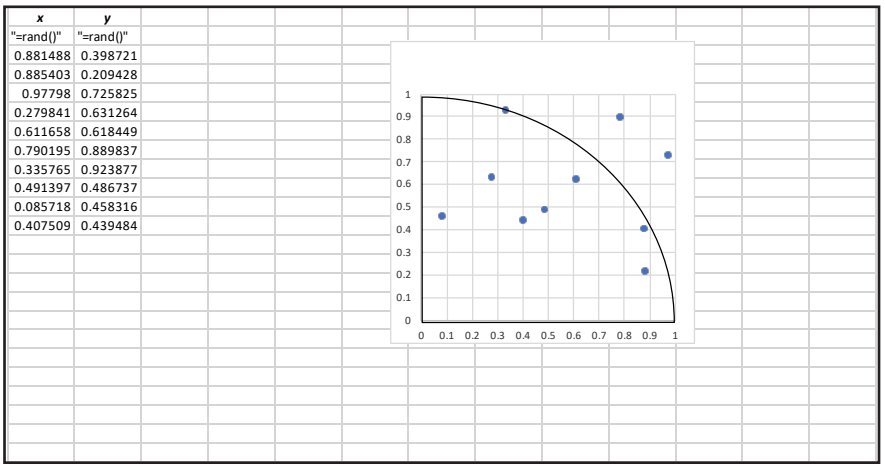
A digital dartboard simulation

Show students slide 5, which displays a spreadsheet containing a quarter circle inside a grid with 10 intervals on each axis. If necessary, explain that the formula '=rand( )' generates random numbers between 0 and 1.

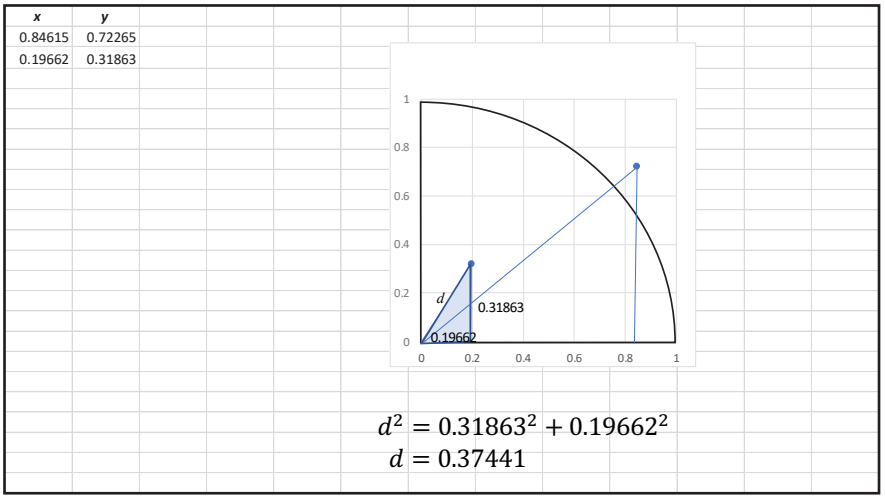


x	y
"=rand()"	"=rand()"
0.881488	0.398721
0.885403	0.209428
0.97798	0.725825
0.279841	0.631264
0.611658	0.618449
0.790195	0.889837
0.335765	0.923877
0.491397	0.486737
0.085718	0.458316
0.407509	0.439484

Ask students to estimate the value of π by observing the number of points inside or outside the circle. Double click inside the spreadsheet image to open it, then use F9 to generate a new set of points and obtain a new estimate for π.

Discuss with students the limitations of using a small number of points and of using a visual method for determining whether a given point lies inside or outside the circle.

Show slide 6, which displays only two points on the grid and a calculation of the distance from the origin to one of the points. Ask students to use Pythagoras' theorem to calculate the distance from the origin to the point outside the circle, at coordinates (0.84615, 0.72265).



x	y
0.84615	0.72265
0.19662	0.31863

$$d^2 = 0.31863^2 + 0.19662^2$$
$$d = 0.37441$$

Discuss with students how their calculations can tell them whether a given point is inside or outside a circle. Suggest that they create their own spreadsheet to randomly generate a large number of points, identify the number of points within the circle, and then use this to generate an estimation of π. Note that as we are using Pythagoras' theorem to determine whether a point lies inside or outside the circle, we do not need to plot the points. The use of Pythagoras' theorem to determine whether a point lies inside or outside the quadrant is, of course, the distance formula in the Cartesian plane where one point is (0, 0).

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Resources: reSolve Excel Spreadsheet *2c Circle Spreadsheet* can also be used.

This spreadsheet shows 300 randomly generated and plotted points, each given a value of 1 (inside the circle) or zero (outside the circle). We can count the number of points inside the circle by summing these values, then use the result to estimate π. These calculations are shown in cells N3 to Q6. Columns E to H are used to colour-code the points according to whether they are inside the circle (the text in these columns is coloured white so as not to draw attention).

Students might also use the programming language Python to estimate π; see https://gist.github.com/louismullie/3769218.

A GeoGebra simulation can be found at https://www.geogebra.org/m/cGvXEE36.

Reflection

Discuss the accuracy of the different methods used.

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